On selecting leaves with disjoint neighborhoods in embedded trees

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Introduction

This work focuses on a **generalization** of a **combinatorial result** by A. Aggarwal, L. Giubas, J. Saxe and P. Shor [DCG 1987].

Given an embedded tree, the goal is to select in **linear time** a **constant fraction** of the leaves.

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Part of an algorithm to construct in deterministic **linear time** the: **Voronoi Diagram of points in convex position**, given the convex hull.

Can also be extended to other Voronoi diagrams with **tree structure**:

- → Farthest point VD, given the convex hull.
- → Update of a VD, after deleting a point.
- → Order-k VD, given the order-(k-1) VD.

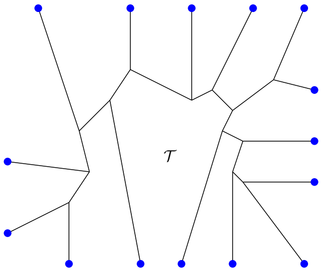
Applications

- The algorithmic scheme has been used to derive linear time algorithms for many problems, e.g.:
  - → **Medial axis** of a simple polygon in $O(n)$.
    [Chin et al. - DCG 1999]
  - → **Order-**k VD in $O(nk^2 + n \log n)$.
    [D.T. Lee - IEEE Trans. Comput. 1982]
  - → **Hamiltonian Abstract** VD in $O(n)$.
    [Klein and Lingas - ISAAC 1994]
  - → **Forest-like Abstract** VD in $O(n)$.
    [Bohler et al. - Comp. Geom. 2014]

Combinatorial result

Theorem [Aggarwal et al. 1987]

Let $\mathcal{T}$ be an embedded binary tree with n leaves



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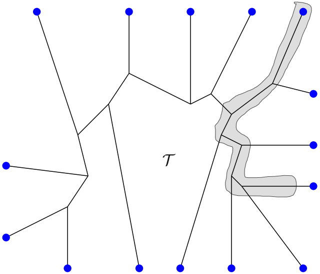

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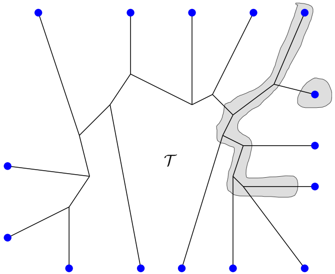

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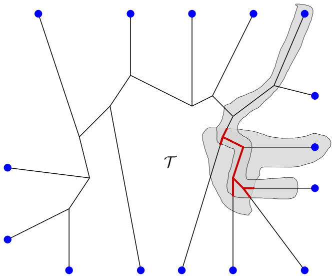

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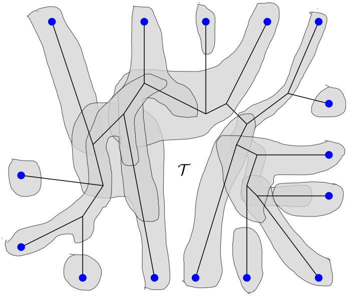

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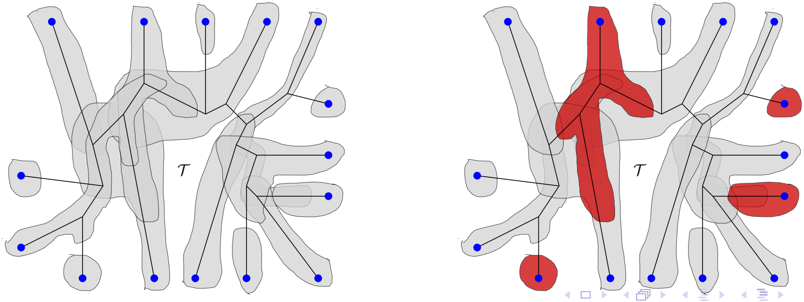

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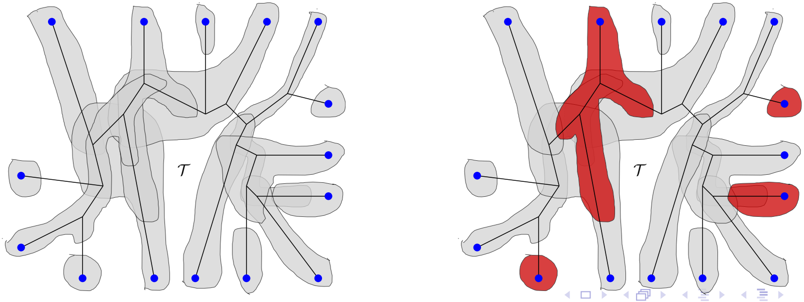

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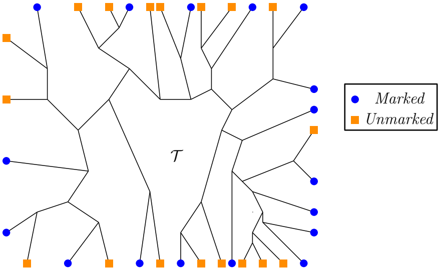

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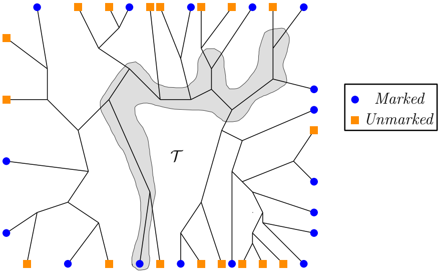

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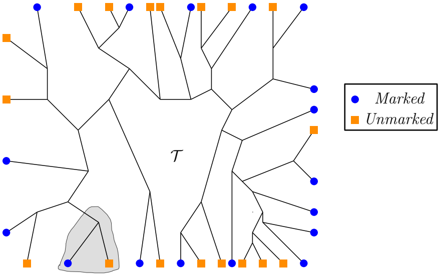

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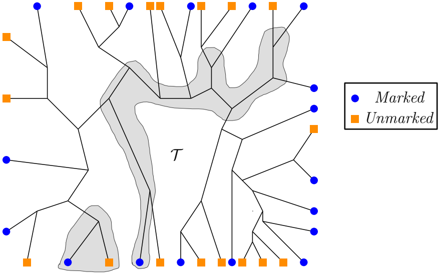

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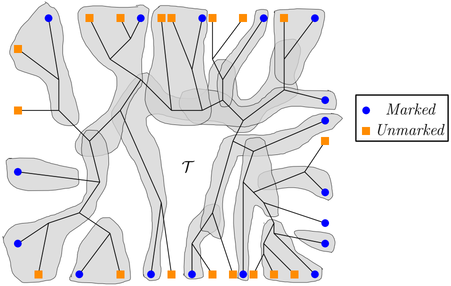

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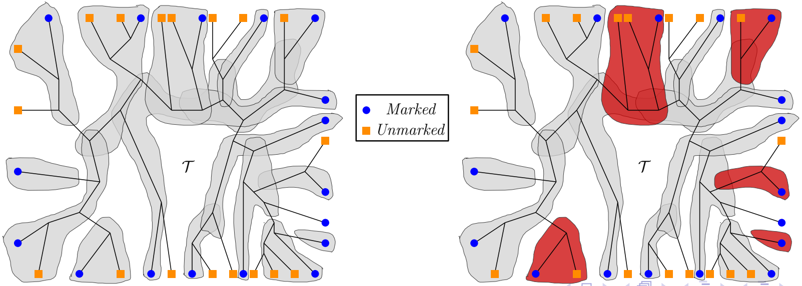

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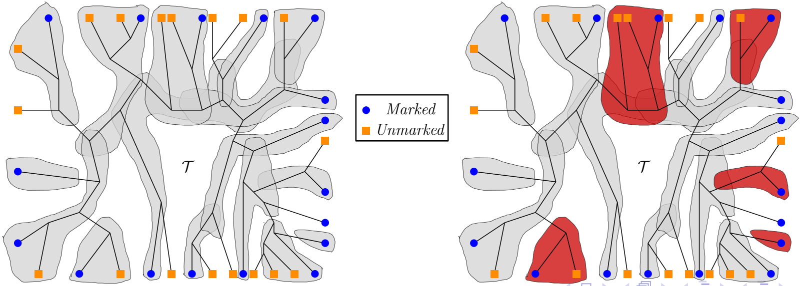

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- If p is a constant, then the algorithm has $O(n)$ time complexity.

Motivation

Linear-time algorithms for problems mentioned *(e.g. deletion of a site, construction of order-k, etc.)* remain open for:

- **Voronoi diagram of non-point sites**
  ...even for simple sites as circles, line segments, etc.
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Suggests that, to potentially apply the linear-time framework...
... We first need this **generalized combinatorial result.**

Outline of results

1. Present some necessary **preliminaries**.
2. Show the first part of the theorem, the **existence**.
3. Show the second part of the theorem, the **algorithm**.

Labeling the nodes [Aggarwal et al. 1987]

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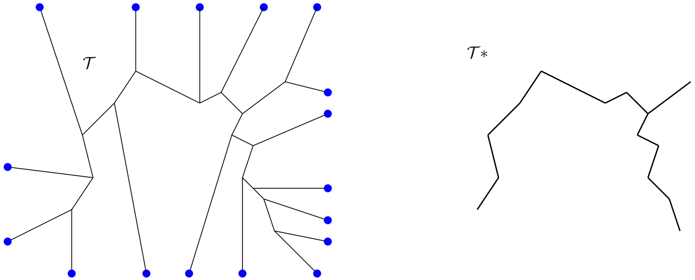

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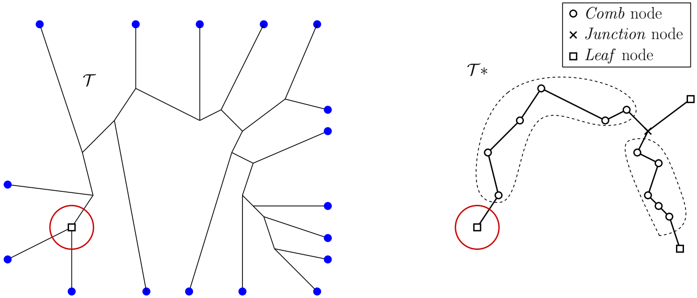

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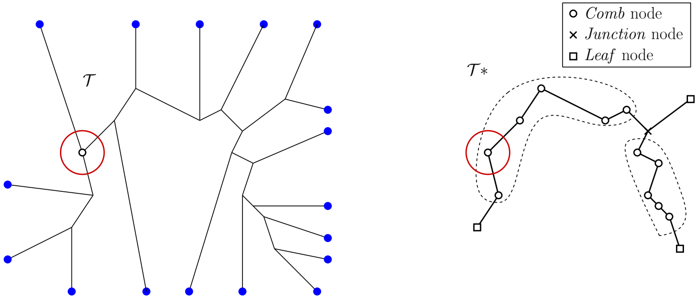

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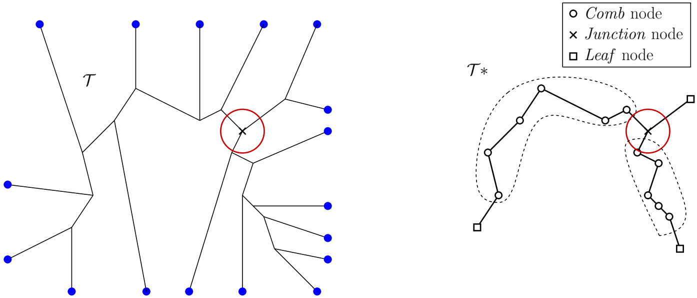

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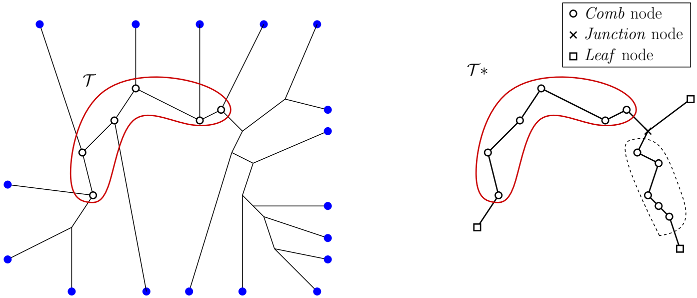

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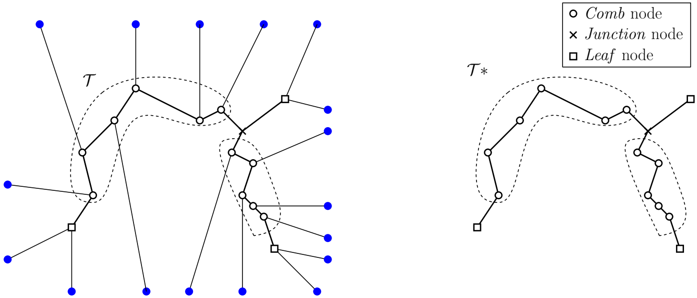

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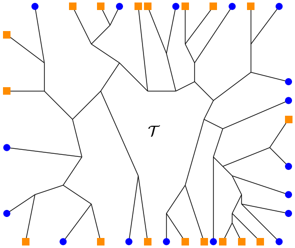

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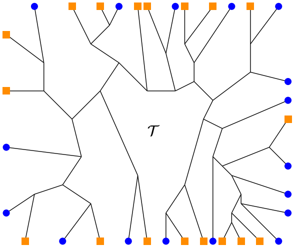

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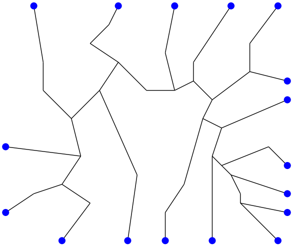

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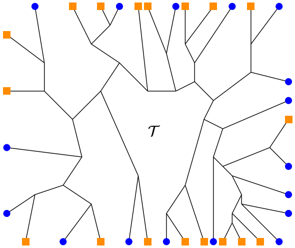

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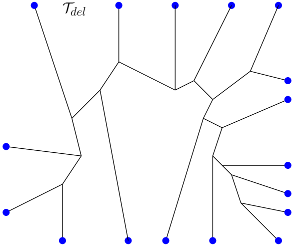

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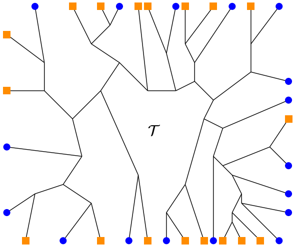

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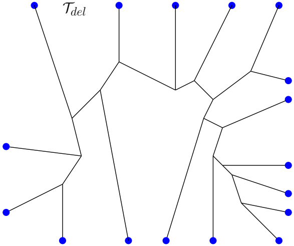

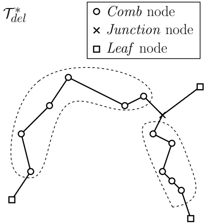

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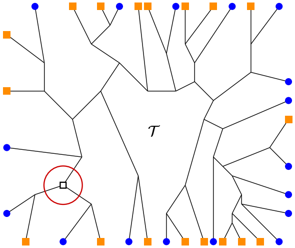

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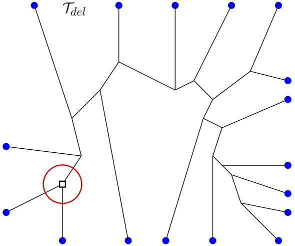

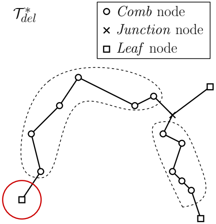

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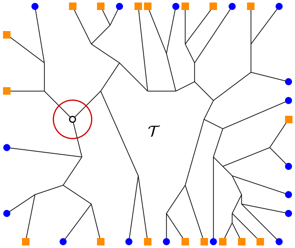

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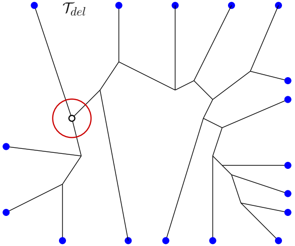

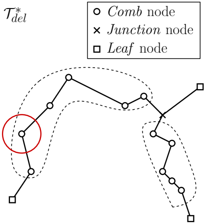

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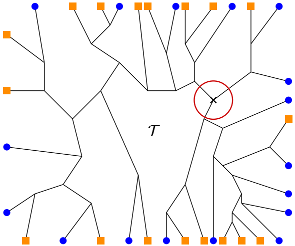

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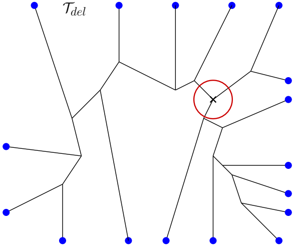

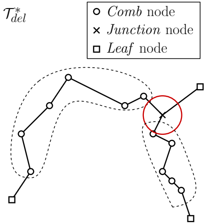

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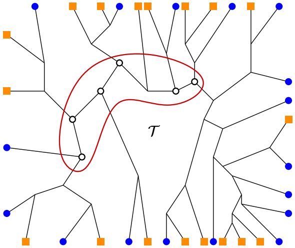

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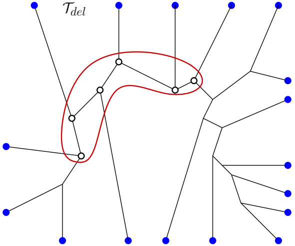
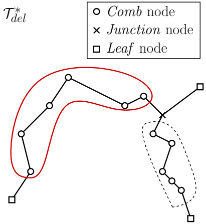
Spine: A sequence of consecutive Comb nodes.





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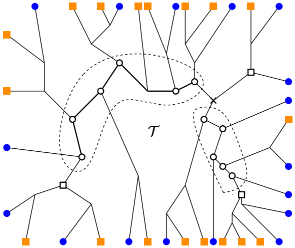
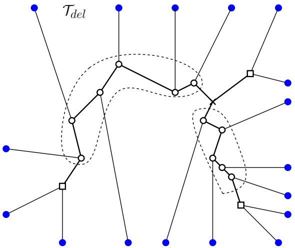
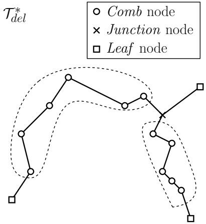


Remark:
Original: **All** internal nodes get labeled.
Generalized: Only **a subset** of the internal nodes get labeled.





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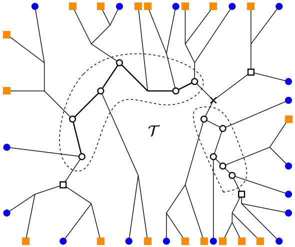

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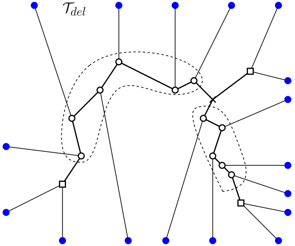
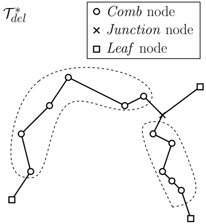
Idea:
Pass the information of the marked leaves to **a subset of $\mathcal{T}$** to resemble [Aggarwal et al. 1987].





Components

Define **two types of components**, which are subtrees of $\mathcal{T}$.

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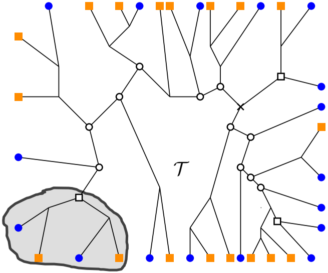
L-component: *The Leaf node and the* 2 *subtrees hanging off that node.*



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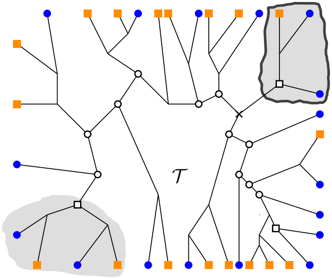

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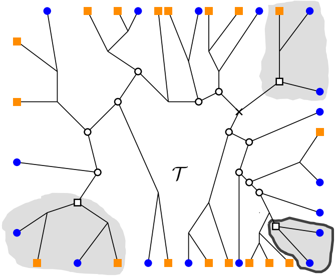

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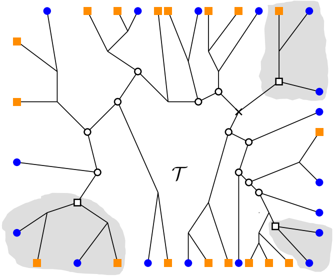

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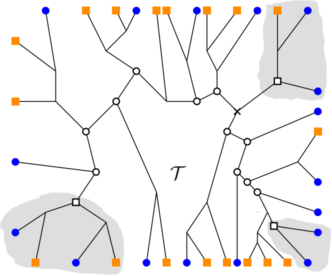
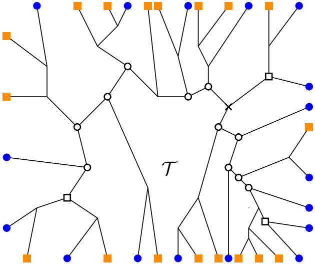
Subdivide each spine into groups of 5 Comb nodes.





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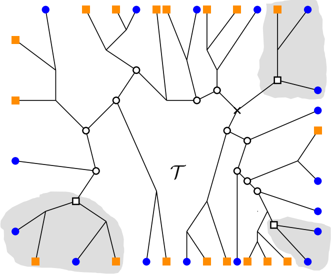

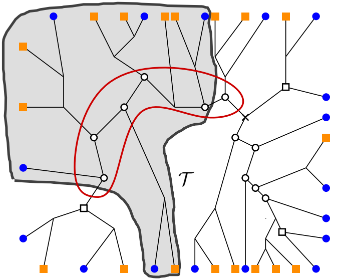

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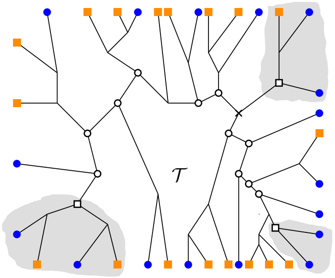

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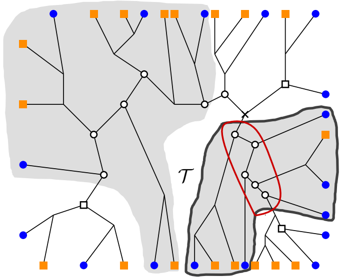

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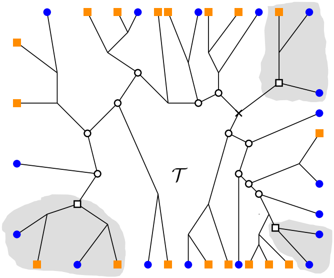

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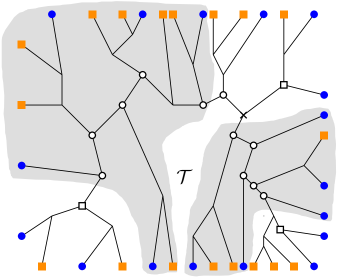

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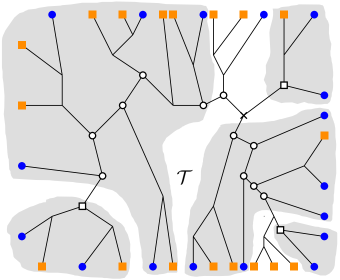

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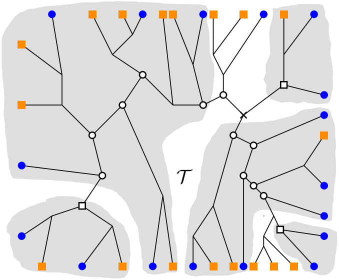

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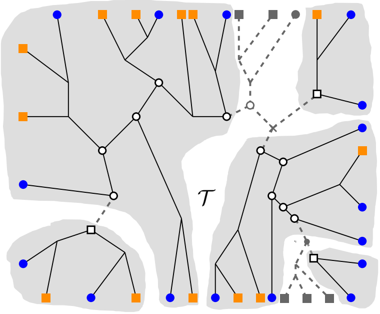

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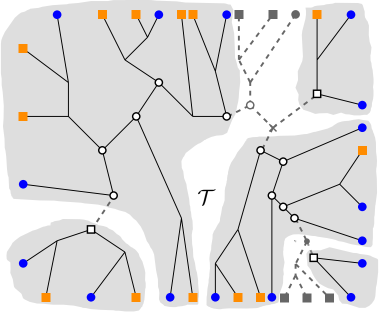

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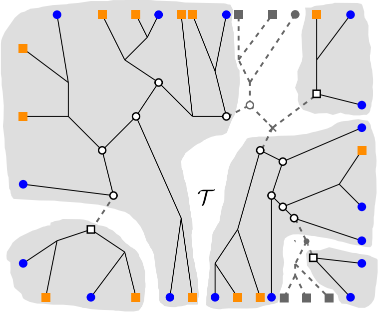

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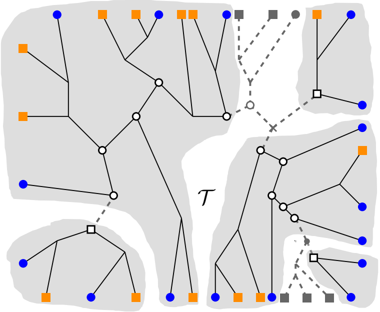

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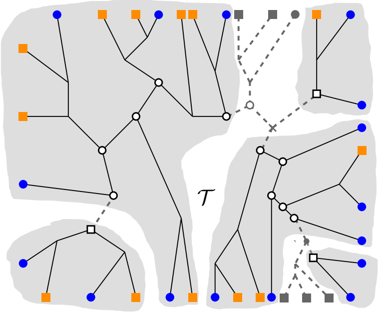

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- Each L-component has 2 marked leaves.
- Each 5-component has 5 marked leaves.
- Not every node belongs to a component.
- A component can have $\Theta(n)$ nodes.

Existence

We want to prove:

Lemma - Existence

At least $\frac{1}{10}m$ marked leaves of $\mathcal{T}$ have pairwise disjoint neighborhoods.

Existence proof

Lemma 1

In every component, there a exists at least one marked leaf ℓ with neighborhood $nh(\ell)$ confined to that component.

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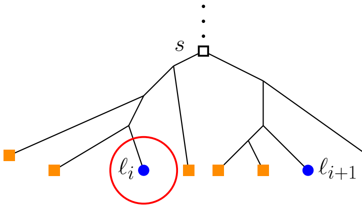

For an **L-component**: consider $nh(\ell_i)$.

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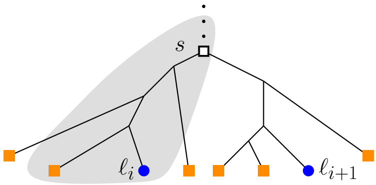

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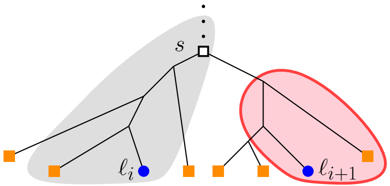

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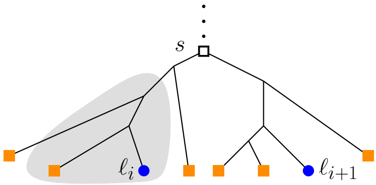

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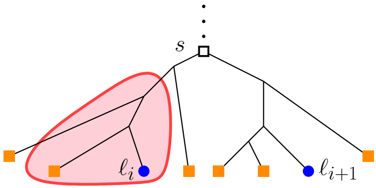

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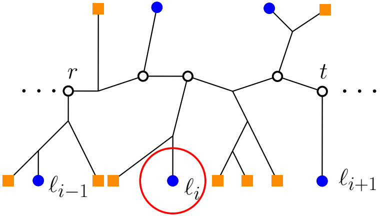

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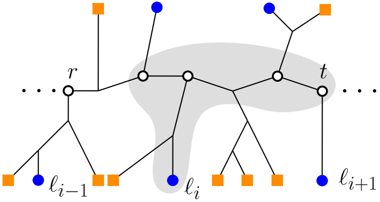

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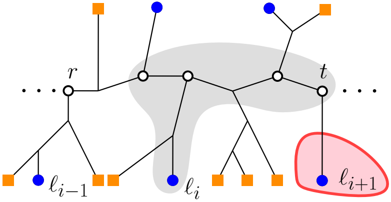

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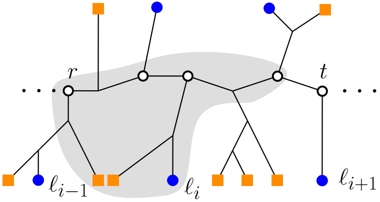

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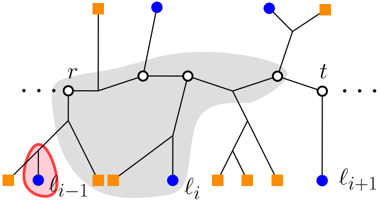

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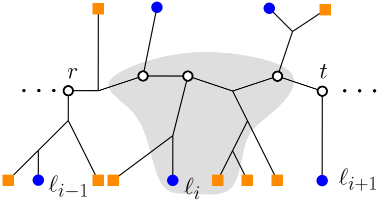

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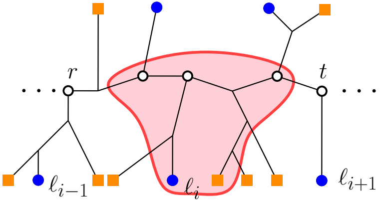

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Corrolary - Lemma 1

The number of marked leaves with a confined neighborhood is:
$\rightarrow$ At least 1 out of 5 in every 5-component.
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Observation

Each spine has at most 4 ungrouped Comb nodes.

Lemma 2

For every 8 ungrouped *Comb* nodes there exists at least 1 L-component.

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Combining the above, we conclude:

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Designing an algorithm

Goal: Design an algorithm to return a fraction of the marked leaves with pairwise disjoint neighborhoods.

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$\rightarrow$ A single neighborhood can require $\Theta(n)$ time to be identified.

Introduce a parameter $p \in (0, 1)$ in the algorithm.
Trade-off between time complexity and number of selected leaves.

Algorithm description

1. Label the tree $\mathcal{T}$ and obtain the components.
2. **For each component** K check up to a **fixed number of steps** $O(z)$:

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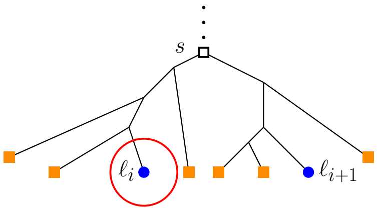

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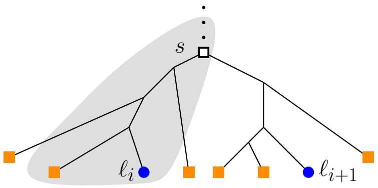

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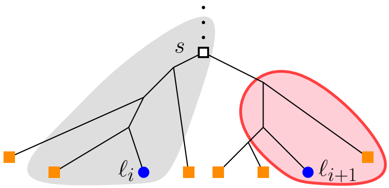

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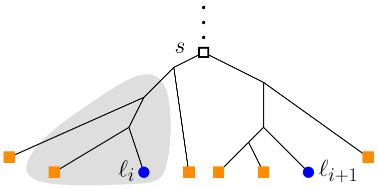

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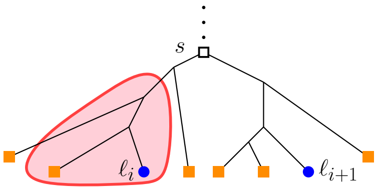

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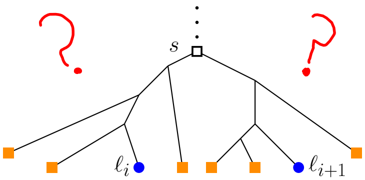

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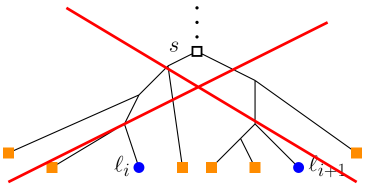

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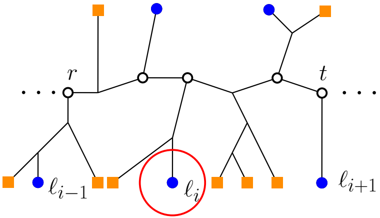

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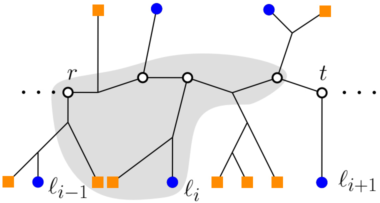

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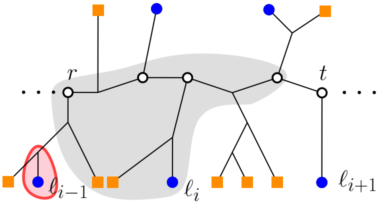

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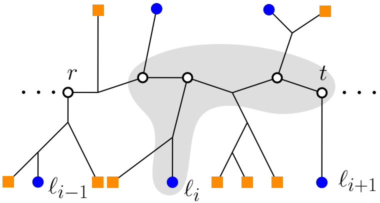

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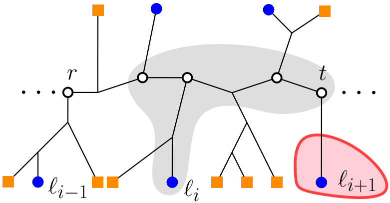

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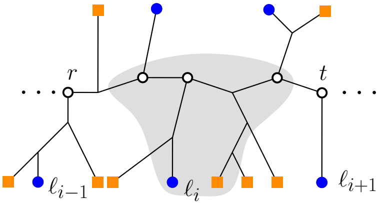

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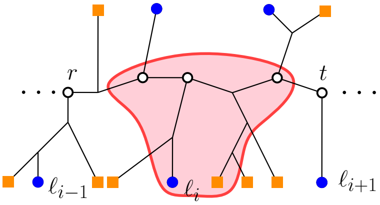

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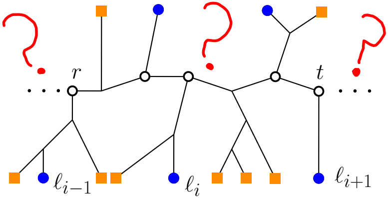

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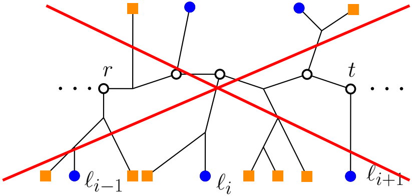

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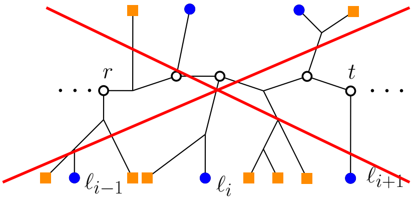

5. **Return** selected leaves.

Algorithm proofs

Need to show:

Algorithm Correctness

Lemma - Correctness

The algorithm returns at least $\frac{p}{10}m$ leaves with pairwise disjoint neighborhoods.

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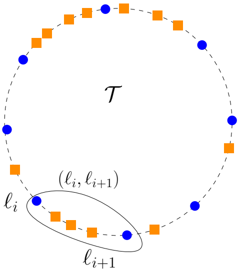
Algorithm time complexity

Lemma -Time complexity

The algorithm has time complexity $O(\frac{1}{1-p}n)$.

Correctness proof

Intervals



Correctness proof

Idea:
Lower bound the number of intervals that do not have *many* unmarked leaves.

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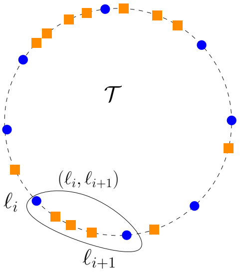

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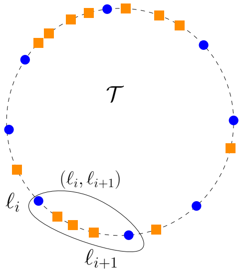

Lemma - *Pigeonhole*

Let M_x be the number of marked leaves whose intervals have at most x unmarked leaves, $x \in \mathbb{N}$. Then $|M_x| \geq \frac{x - c + 1}{x + 1} m$ holds.

c is the ratio between unmarked and marked leaves, $c = \lceil \frac{n-m}{m} \rceil$.

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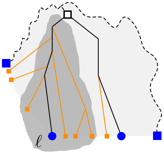
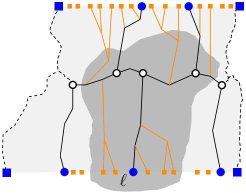
Idea:
Upper bound the size of a confined neighborhood by the number of unmarked leaves in the intervals related to the component.





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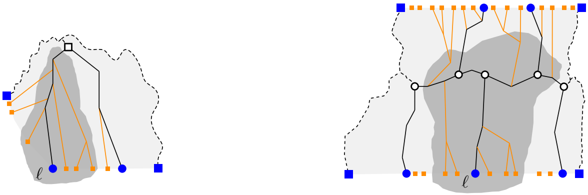

Lemma - Size of confined neighborhoods

Let K be component and a marked leaf ℓ with neighborhood $nh(\ell)$ confined K. Then, $|nh(\ell)| < 10\delta_K$.

δ_K is the maximum size of intervals related to the component K.

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Lemma -Time complexity

The algorithm has time complexity $O(\frac{1}{1-p}n)$.

- There are $\Theta(m)$ components.

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- There are $\Theta(m)$ components.
- For each component, the algorithm does a fixed number of steps ($\leq 10z$).
  By using $z = \left\lceil \frac{10c}{1-p} \right\rceil = \Theta(\frac{c}{1-p})$, the claim follows.

Conclusion

Theorem - Generalized

Let $\mathcal{T}$ be an embedded binary tree with n leaves where:
i) m of the leaves have been marked.
ii) Each marked leaf of $\mathcal{T}$ has a neighborhood.
iii) Topologically consecutive marked leaves have disjoint neighborhoods.
Then:
i) $\exists \geq \frac{1}{10}m$ marked leaves with pairwise disjoint neighborhoods.
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Expect it to be helpful in designing deterministic **linear time algorithms** for problems related to **abstract Voronoi diagrams** and other **generalized Voronoi diagrams**.

Thank you for your attention!