Linear-size farthest color Voronoi diagrams: conditions and algorithms

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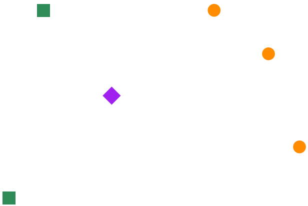
Color Voronoi Diagrams

- Family $\mathcal{P}$ of m **clusters** (sets) of points, with n total points.
  $\rightarrow$ Each cluster has a different color.



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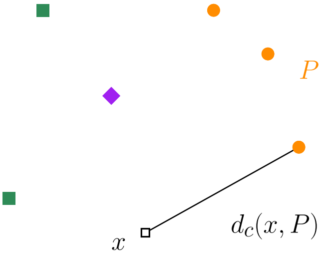

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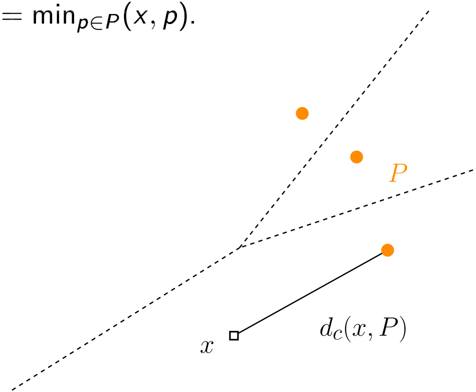
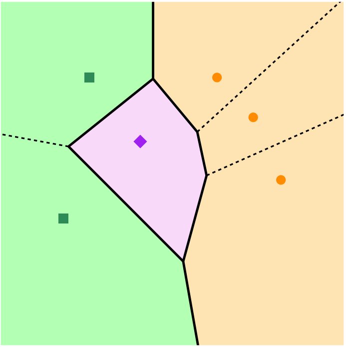


Nearest Color Voronoi Diagram (NCVD)

- The **nearest color region** of a cluster $P \in \mathcal{P}$ is: $\{x \in \mathbb{R}^2 \mid d_c(x, P) < d_c(x, Q),\ \forall Q \in \mathcal{P} \setminus \{P\}\}$



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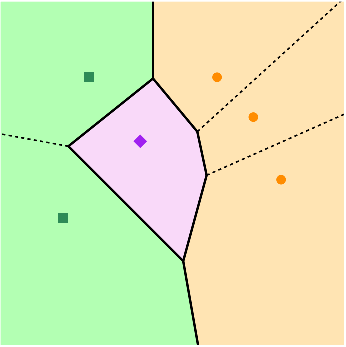

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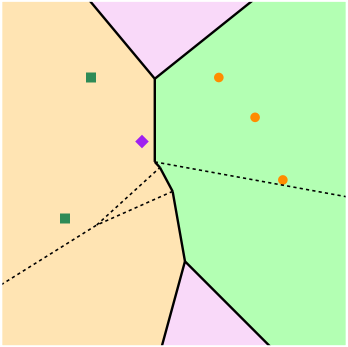

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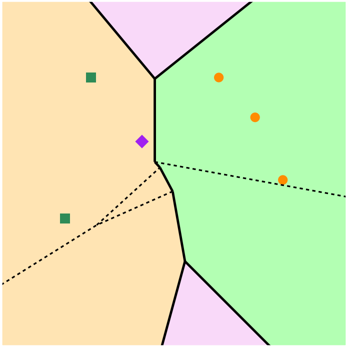

FCVD History

- Construction algorithm $O(mn \log n)$.
  Worst case complexity $\Omega(mn)$ - $O(mn\alpha(mn))$.
  [Huttenlocher, Kedem and Sharir 1993]

$m = |\mathcal{P}|$
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- Special cases.
  [Bae 2012, Claverol et al. 2017, Iacono et al. 2017]

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Motivation - Applications

- **Minimum Hausdorff distance** between two sets of points. [Huttenlocher et al. 1993]
- **Facility location** with multiple types of facilities. [Abellanas et al. 2006]
- Euclidean Bottleneck **Steiner tree**. [Bae et al. 2010]
- **Sensor deployment** in wireless networks. [Lee et al. 2010]
- **Stabbing circles** for segments. [Claverol et al. - 2017]

FCVD relation to Hausdorff Voronoi Diagram

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  - → Envelopes in 3 dimensions [Edelsbrunner et al. 1989]
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- FCVD can be computed in $O(n^2)$, by adapting the algorithm of [Edelsbrunner et al. 1989].

Summary of results

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- **Sufficient conditions** for FCVD to have $O(n)$ combinatorial complexity.
- **Construction algorithms** when these condition are met.

Color bisectors

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  $b_c(P, Q) = \{x \in \mathbb{R}^2 \mid d_c(x, P) = d_c(x, Q)\}$

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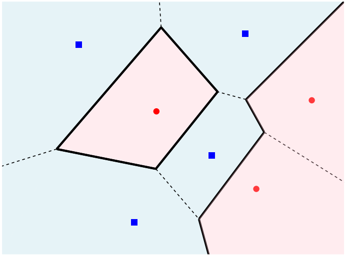

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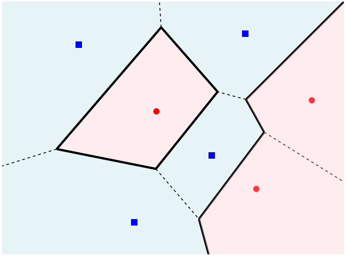

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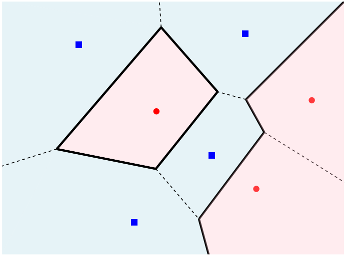
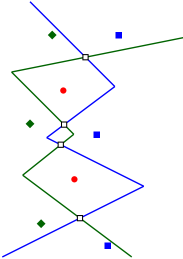


→ It consists of cycles and unbounded chains.

→ Two bisectors may intersect linearly many times.



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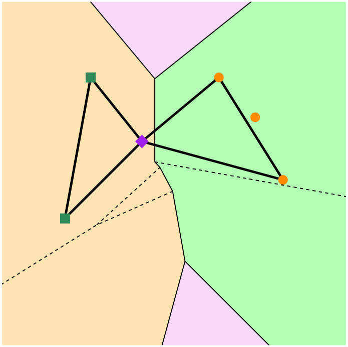

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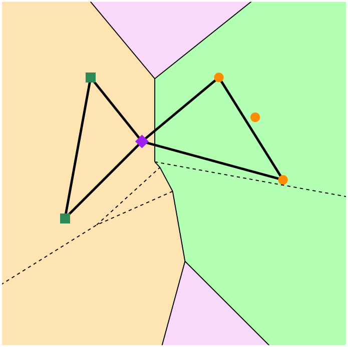

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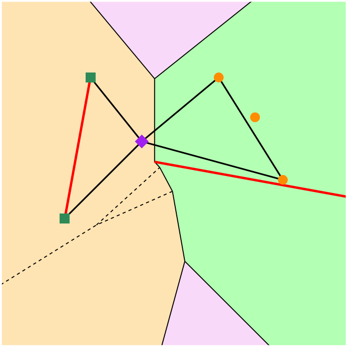

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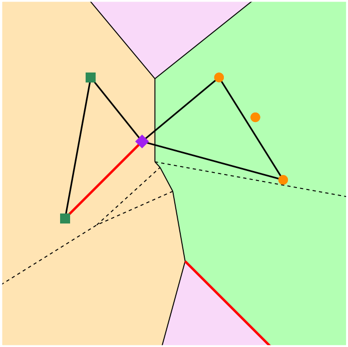

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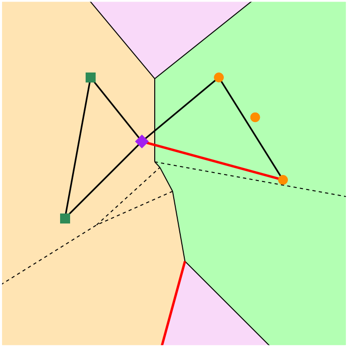

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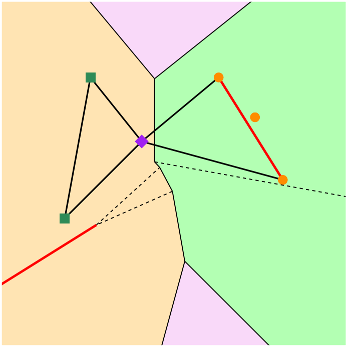

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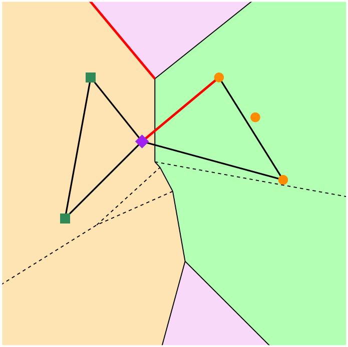

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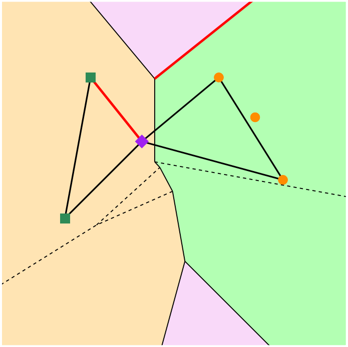

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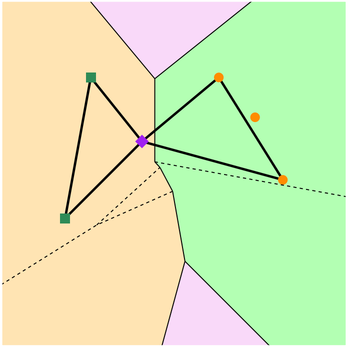

Abstract Voronoi diagrams

Abstract Voronoi diagrams [Klein 1989]

Voronoi diagrams are **defined on a system of a bisectors** that satisfy a set of axioms. For every $\mathcal{P}' \subseteq \mathcal{P}$:

$A1$. Each bisector is an unbounded Jordan curve.

$A2$. Each nearest neighbor region is non-empty and connected.

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A family of clusters $\mathcal{P}$ is called **admissible**, if the color bisectors satisfy axioms A1-A3.

Admissible families

Proposition - Structure and complexity

If $\mathcal{P}$ is an admissible family, then FCVD($\mathcal{P}$) is a tree of $O(n)$ total combinatorial complexity.

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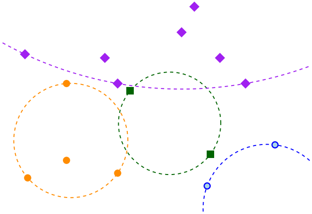
Corollary - Admissible check

We can check if a family $\mathcal{P}$ is admissible in $O(n \log n)$ time.

Disk-separable families

A family of clusters $\mathcal{P}$ is **disk-separable** if for every $P \in \mathcal{P}$ there exists a disk containing P and no point from other cluster.



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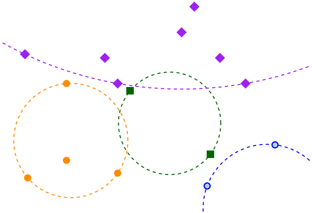
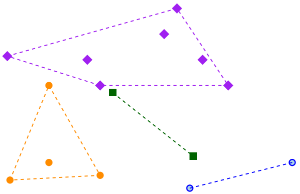


Proposition - Sufficient condition

If a family $\mathcal{P}$ is disk-separable, then $\mathcal{P}$ is also admissible.

Linearly-separable families

A family of clusters $\mathcal{P}$ is **linearly - separable** if all clusters have pairwise disjoint convex hulls.



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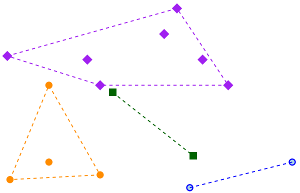

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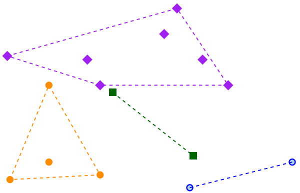

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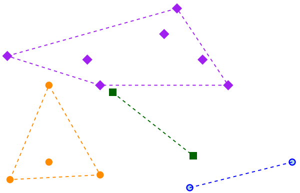

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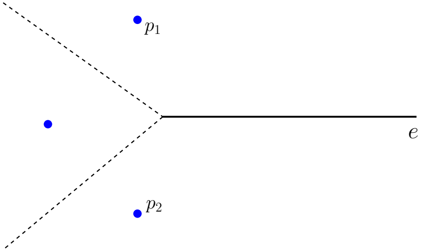

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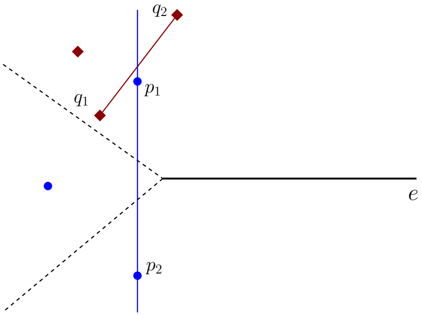

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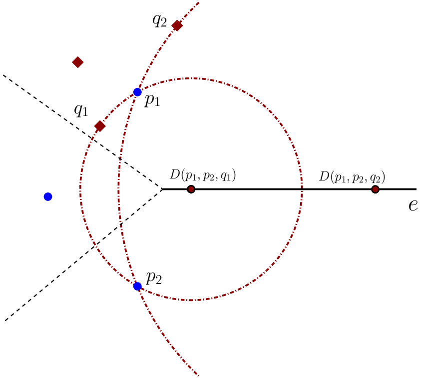

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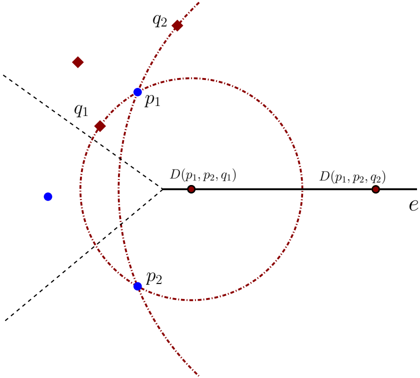

The number of all clusters that straddle e is $s(e)$.

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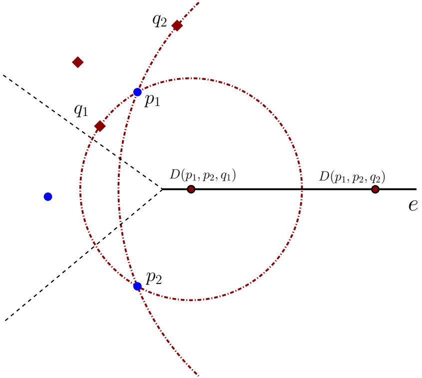

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Corollary - Condition

If a linearly-separable family $\mathcal{P}$ has $s(\mathcal{P}) = O(n)$, then FCVD$(\mathcal{P})$ has $O(n)$ combinatorial complexity.

Algorithm description

Divide & Conquer algorithm

1. **Split** family of clusters $\mathcal{P}$ in two parts $\mathcal{P}_L$, $\mathcal{P}_R$.
2. **Recursively compute** FCVD($\mathcal{P}_L$) and FCVD($\mathcal{P}_R$)
3. **Merge** FCVD($\mathcal{P}_L$), FCVD($\mathcal{P}_R$) into FCVD($\mathcal{P}$).

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The **merge curve** consists of bounded & unbounded components.
For each component:

a. **Find a starting point**.
b. **Trace the component**.

Merge curve construction

a. **Finding starting points.**

b. **Tracing components.**

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a. **Finding starting points.**
   - $\rightarrow$ Unbounded components can be found in $O(n)$ time using the cluster hull.
   - $\rightarrow$ Bounded components ???

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$\rightarrow$ A component M can be traced in $O(|M|)$ time.

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Key: FCVD($\mathcal{P}$) is a tree, so there are no bounded components.

Algorithms

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Key: The bounded components can be found in $O(n \log n)$ time [Iacono et al. 2017].

Future work

- **Settle FCVD complexity** of linearly-separable families.
  $\rightarrow$ We conjecture it is $\Theta(mn)$ in the worst case.

- **Design** $o(n^2)$ **algorithm** when FCVD has $O(n)$ complexity.

Thank you for your attention!