Farthest Color Voronoi diagrams: conditions and algorithms

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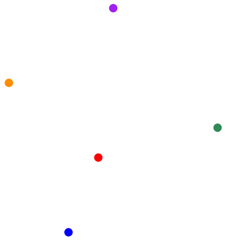
Point Voronoi Diagrams

- $\mathcal{P}$: A set of points in $\mathbb{R}^2$



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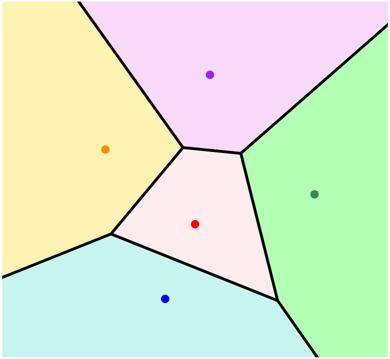

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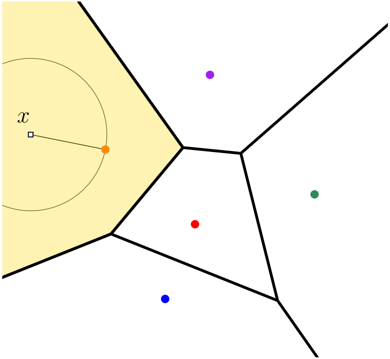

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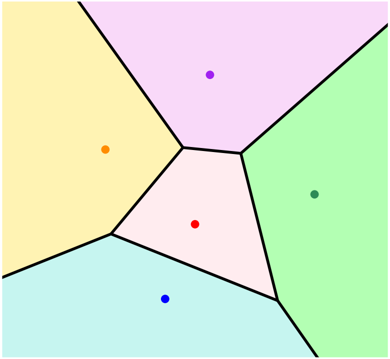

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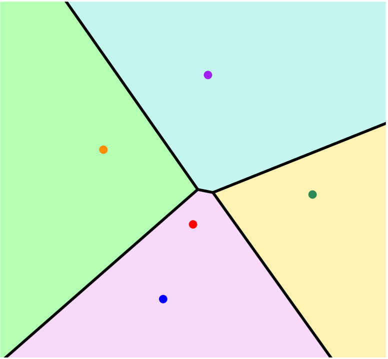

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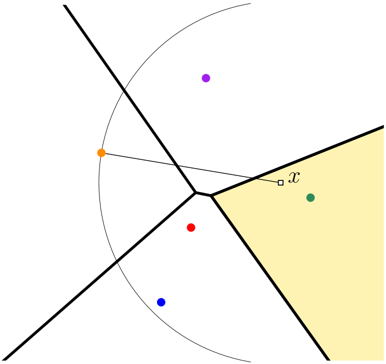

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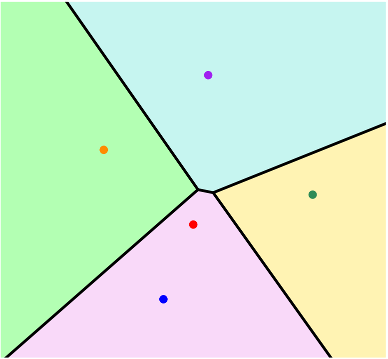

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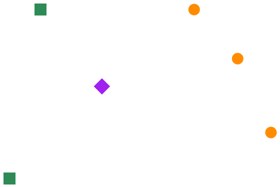

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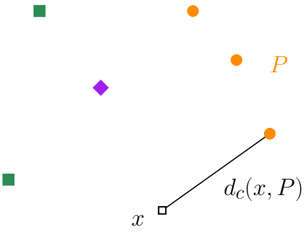

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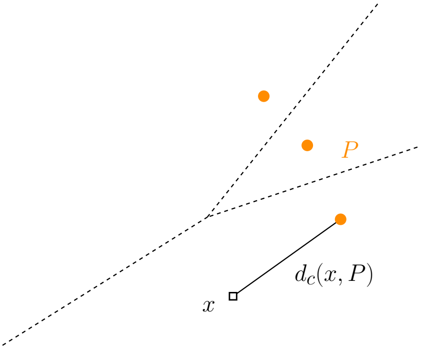

Nearest Color Voronoi Diagram (NCVD)

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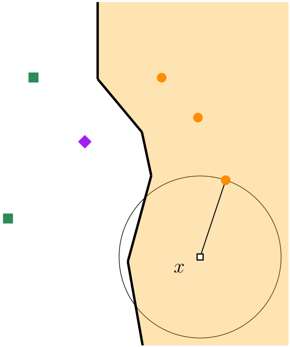

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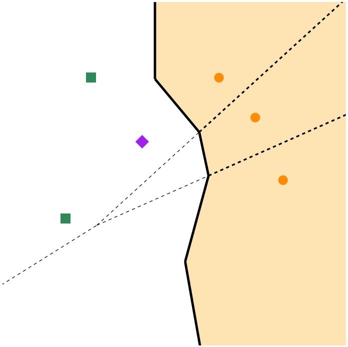

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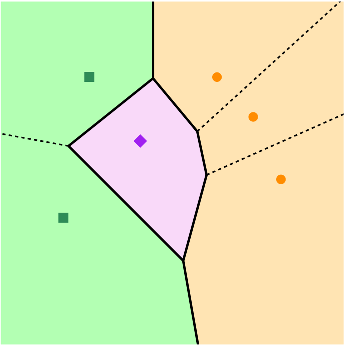

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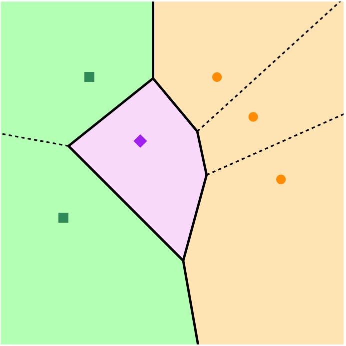
- Nearest point VD $\Rightarrow$ $O(n)$ complexity, $O(n \log n)$ algorithms.



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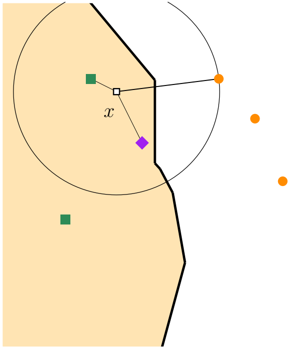

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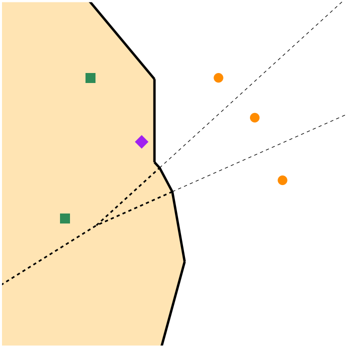

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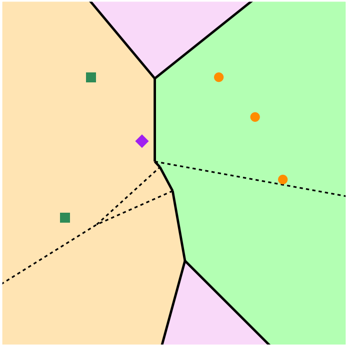

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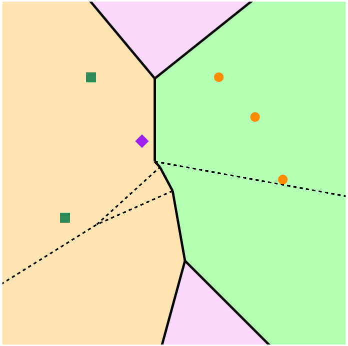

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- Combinatorial **complexity**:
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Motivation - Applications

- **Facility location** with multiple types of facilities.
  Minimum color spanning circle. [Abellanas et al. 2006]
- **Minimum Hausdorff distance** between two sets of points. [Huttenlocher et al. 1993]
- Euclidean Bottleneck **Steiner tree**. [Bae et al. 2010]
- **Sensor deployment** in wireless networks. [Lee et al. 2010]
- **Stabbing circles** for segments. [Claverol et al. 2017]

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  Extensively studied:

  → Envelopes in 3 dimensions [Edelsbrunner et al. 1989]
  → Divide and Conquer [Papadopoulou & Lee 2004]
  → Plane Sweep [Papadopoulou 2004]
  → Randomized Incremental [Arseneva & Papadopoulou 2019]

Color bisectors

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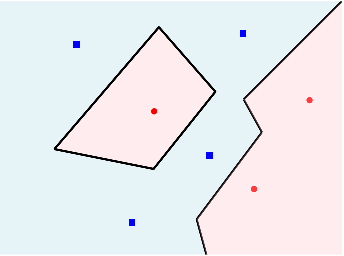

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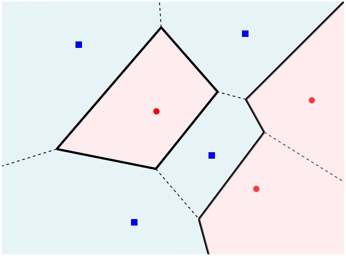

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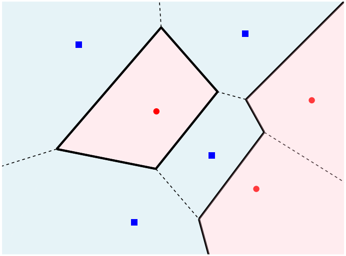

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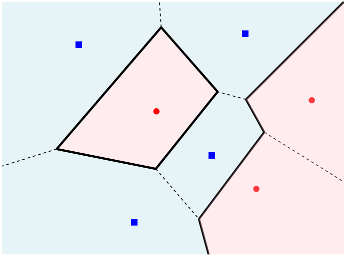
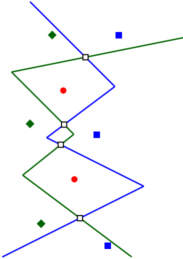


→ It consists of bounded and unbounded components.

→ Two bisectors may intersect linearly many times.



Cluster Hull

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- $O(n)$ size - $O(n \log n)$ time construction.

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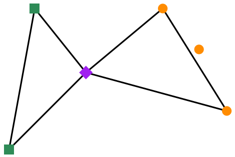

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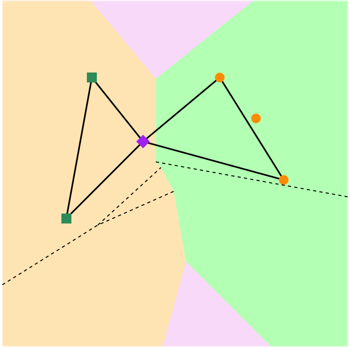

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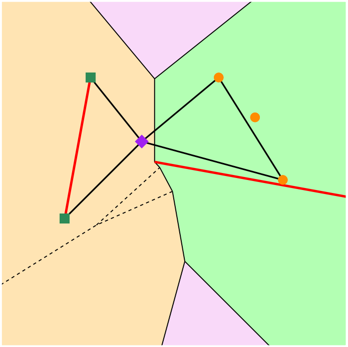

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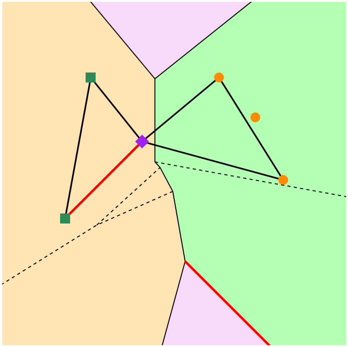

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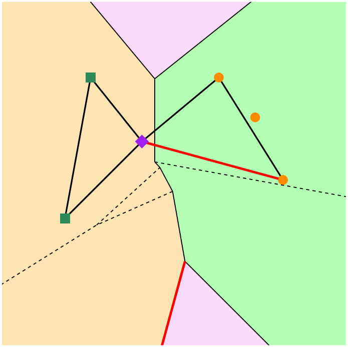

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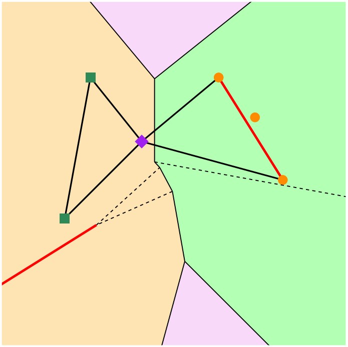

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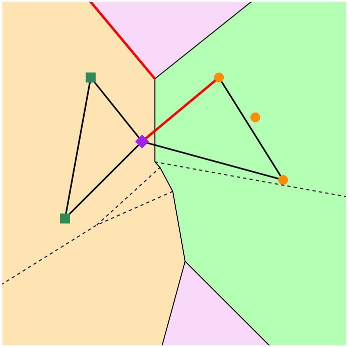

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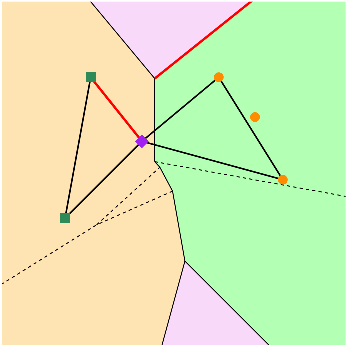

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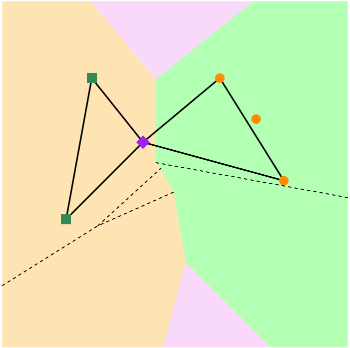

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- **Bounded** faces?

  Key: Define **straddles**.

Straddles

Cluster Q (or $q_1, q_2 \in Q$), **straddles** $p_1, p_2 \in P$ if:
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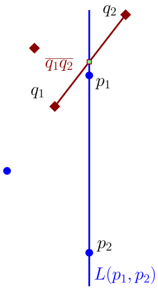

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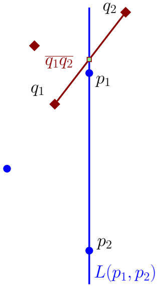

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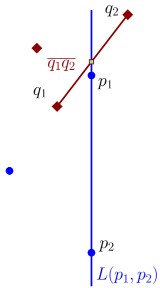

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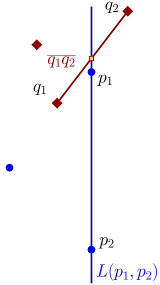

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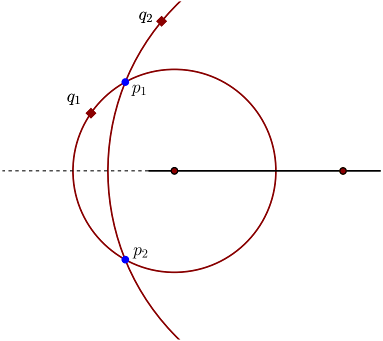
$\rightarrow \overline{q_1 q_2}$ intersects $L(p_1, p_2)$.

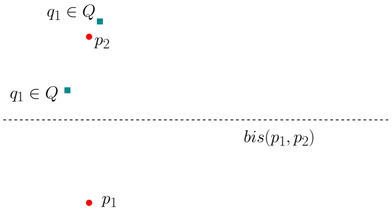
$s(p_1, p_2)$: number of clusters that straddle p_1, p_2.



- $s(\mathcal{P})$: **straddling number** of $\mathcal{P}$
- $s(\mathcal{P}) = \sum_{P_i \in \mathcal{P}} \sum_{(p_j, p_k) \in P_i} s(p_j, p_k)$
- $s(\mathcal{P}) = O(mn)$

Refined combinatorial complexity

- FCVD($\mathcal{P}$) has $O(n + s(\mathcal{P}))$ bounded faces.



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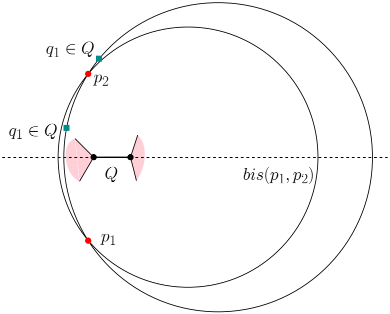

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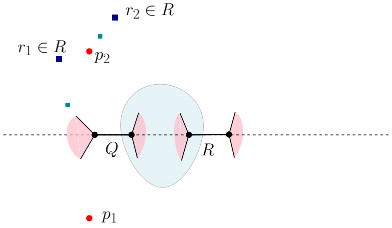

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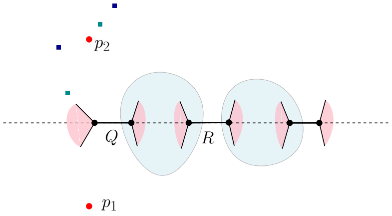

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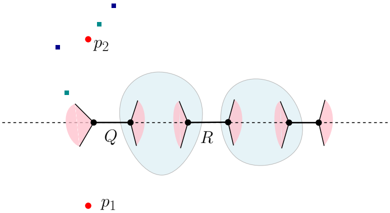

Theorem - Combinatorial complexity

FCVD($\mathcal{P}$) has $O(n + s(\mathcal{P}))$ complexity.

Note: Refine $O(mn)$ upper bound [Abellanas et al. 2006].

Conditions for linear-size diagrams

If FCVD($\mathcal{P}$) has $\Theta(n^2)$ size, the $O(n^2)$ algorithm is optimal.

We are interested in:

- **Conditions for $O(n)$ combinatorial complexity.**
- $o(n^2)$-**time algorithms** for these cases.

Admissible cluster 1/2

$\mathcal{P}$ is **admissible**, if it falls under the framework of **Abstract** Voronoi diagrams [Klein 1989].

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- If $\mathcal{P}$ is admissible, FCVD($\mathcal{P}$) is a tree of $O(n)$ complexity. Follows [Mehlhorn et al. 2001]

Admissible cluster 2/2

Theorem - Necessary & sufficient condition

$\mathcal{P}$ is admissible if and only if:
(1) each region in NCVD($\mathcal{P}$) is connected.
(2) no cluster is contained in the convex hull of another cluster.

Key: Check region connectivity **only for** $\mathcal{P}$.

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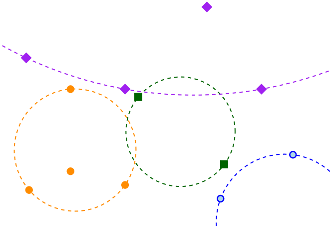
Key: Check region connectivity **only for** $\mathcal{P}$.

- If $\mathcal{P}$ is linearly separable, we can decide if $\mathcal{P}$ is admissible in $O(n \log n)$ time.

  Key: Check region connectivity using NCVD($\mathcal{P}$).

Disk-separable clusters

$\mathcal{P}$ is **disk-separable** if:
for any $P \in \mathcal{P}$ there exists a disk containing P and no other point.



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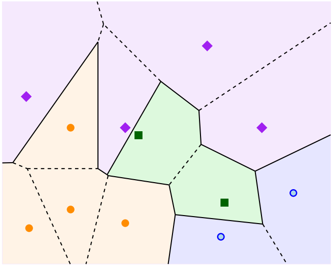


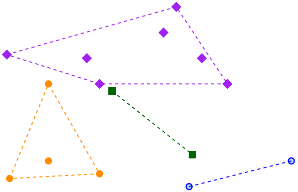
Theorem - Sufficient condition

If $\mathcal{P}$ is disk-separable, then $\mathcal{P}$ is admissible.

Key: Disk-separability implies region connectivity.

Linearly separable clusters

$\mathcal{P}$ is **linearly separable** if:
clusters have pairwise disjoint convex hulls.



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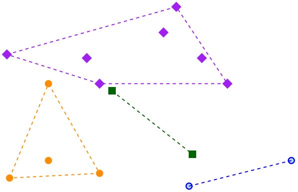

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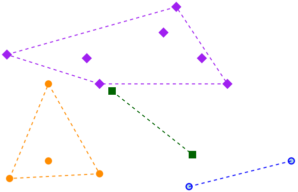

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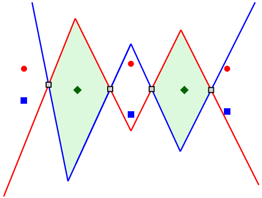

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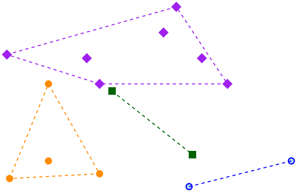

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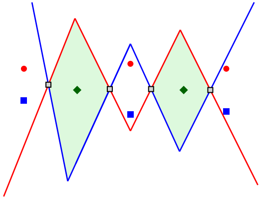

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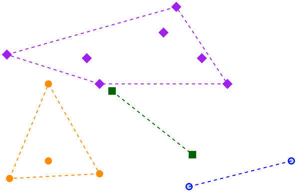

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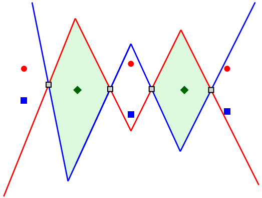

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We construct a family $\mathcal{P} = \{P_i = \{l_i, u_i\}\}$.
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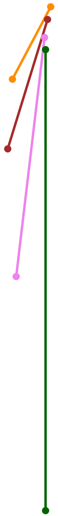

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- Halve length and translate upwards.
- Perform some *"rotation"* around upper points

Lower bound Construction 2/2

Properties of the constructed set $\mathcal{P}$

- $\mathcal{P}$ is linearly separable
- $s(\mathcal{P}) = \Theta(m^2)$
- Every straddle induces a vertex to FCVD($\mathcal{P}$).

Lower bound Construction 2/2

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Combining $\Omega(m^2)$ with trivial $\Omega(n)$ bound:

Theorem - Lower bound

If $\mathcal{P}$ is linearly separable, FCVD($\mathcal{P}$) has $\Omega(n + m^2)$ complexity in the worst case.

Algorithm description

Divide & Conquer algorithm:

1. **Divide** $\mathcal{P}$ in two sets $\mathcal{P}_A$ and $\mathcal{P}_B$.
2. **Recursively compute** FCVD($\mathcal{P}_A$) and FCVD($\mathcal{P}_B$).
3. **Merge** FCVD($\mathcal{P}_A$) and FCVD($\mathcal{P}_B$) into FCVD($\mathcal{P}_A \cup \mathcal{P}_B$).

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Construct the **merge curve**.

$\rightarrow$ Can have many components.

$\rightarrow$ Can be bounded and unbounded.

Constructing the merge curve

For each component:

a. Find a **starting point**.

b. **Trace** the component.

Constructing the merge curve

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a. Find a **starting point**.

b. **Trace** the component.

- **Tracing** a component takes linear time.

  Key: Using a *visibility property*

Constructing the merge curve

For each unbounded component:

a. **Find a starting point.**

b. **Trace** the component.

- Finding **starting points** on **unbounded** components takes $O(n)$ time at each step.

  Key: Merging cluster hulls before merging diagrams, similar to [Papadopoulou & Lee 2004].

Starting points on bounded components

For each bounded component:

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Key: The internal subdivision of every bounded face is a tree.
Need to search for edges of the internal subdivision.

Starting points on bounded components

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Need to search for edges of the internal subdivision.

- $\rightarrow$ Use data structure of [Iacono et al. 2017]
- $\rightarrow$ $O(n \log n)$ time to build at each step.
- $\rightarrow$ For each occurrence of an edge: $O(\log^2 n)$ search procedure.

Algorithm: General case

Theorem - General algorithm

FCVD($\mathcal{P}$) can be constructed in $O((n + s(\mathcal{P})) \log^3 n)$ time.

Key: Make $O(\log^2 n)$ search only for potential bounded faces.
Note: Faster than existing algorithms, if $s(\mathcal{P}) = O(n)$.

Algorithm: Admissible clusters

Theorem - Admissible clusters

If $\mathcal{P}$ is admissible, FCVD($\mathcal{P}$) can be constructed in $O(n \log n)$ time.

Key: FCVD($\mathcal{P}$) is a tree, so no bounded components.

Conclusions

Farthest Color Voronoi Diagrams

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- Refined **combinatorial complexity**: **straddles**.
- **Conditions** under which FCVD has **linear complexity**.
- **Linear separability**: quadratic **lower bound**.
- **Construction algorithms**: $O((n + s(\mathcal{P})) \log^3 n)$.

Thank you for your attention!

Questions?