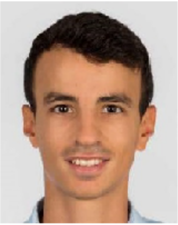
New Variants of
Perfect Non-crossing Matchings

Ioannis Mantas[1] Marko Savić[2] Hendrik Schrezenmaier[3]

[1] Università della Svizzera italiana, Lugano, Switzerland
[2] University of Novi Sad, Serbia
[3] Technische Universität Berlin, Germany

CALDAM 2021



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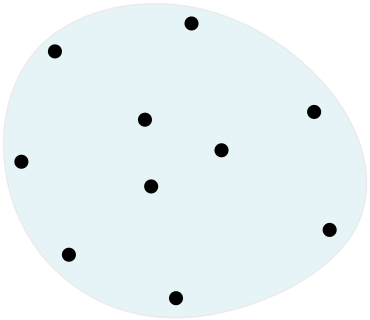

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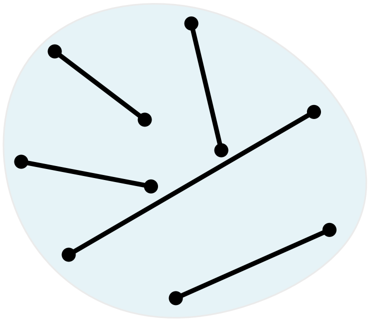

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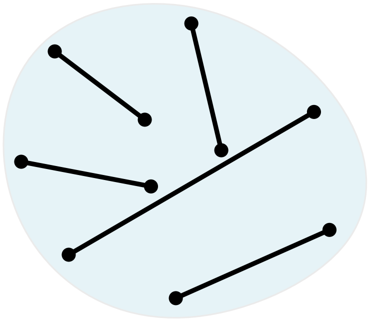

- Perfect: All points are matched
- Non-crossing: No two edges of the matching intersect

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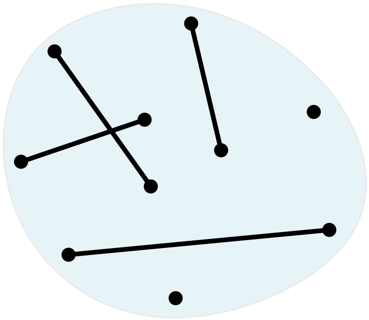

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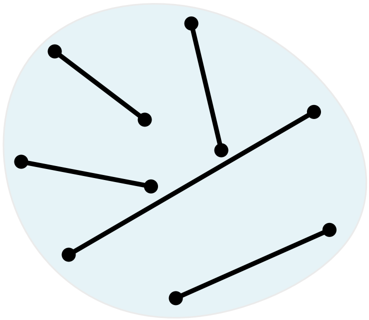
Perfect Non-crossing Matchings

Monochromatic



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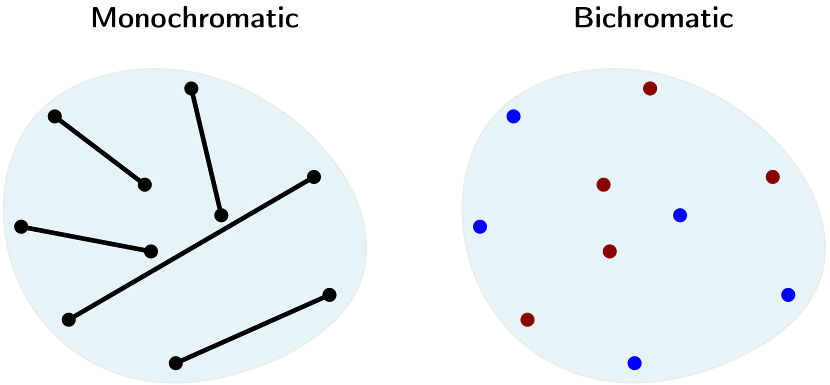

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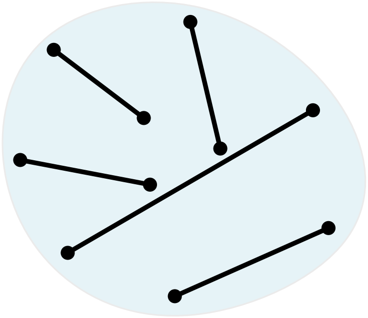
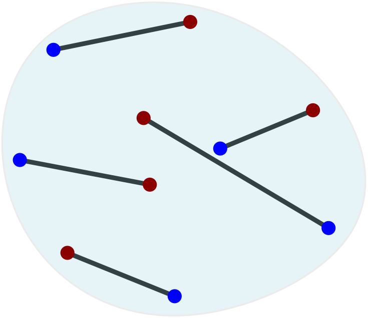


Bichromatic



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Motivation

Some applications:

- Operations Research [Assignment problem]
- VLSI design [Cong et al. 1993]
- Computational Biology [Colannino et al. 2006]
- Map construction/comparison [Eppstein et al. 2015]

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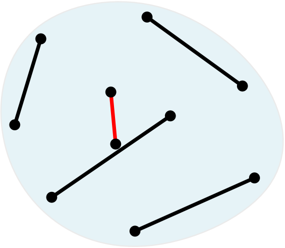
Computation:

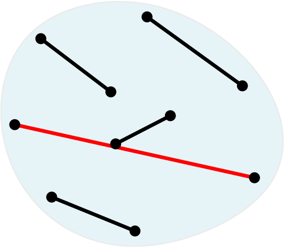
- Arbitrary matching: $O(n \log n)$ time [Hershberger & Suri 1992]
- Often arbitrary not sufficient... optimization criterion

Optimization of Perfect Non-crossing matchings

MinMin



MaxMax



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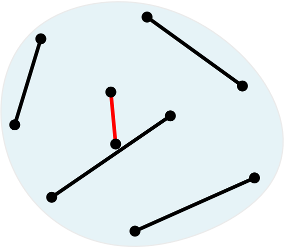

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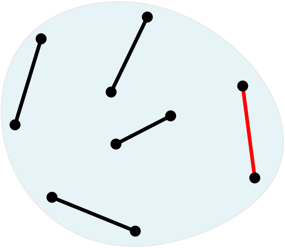

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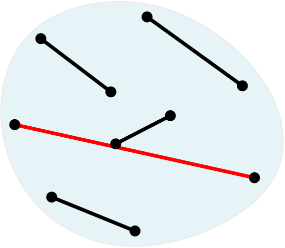

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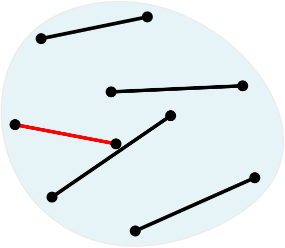

Known Results

Monochromatic	MinMax	
General Position	$\mathcal{NP}$-hard	[Abu-Affash et al. 2014]
Convex Position	$O(n^2)$	[Savić & Stojaković 2017]
Points on circle	$O(n)$	[Savić & Stojaković 2017]
Bichromatic	**MinMax**	
General Position	$\mathcal{NP}$-hard	[Carlsson et al. 2015]
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MinSum (Euclidean Assignment problem)

- Gen. Pos. Monochromatic: $O(n^{1.5} \log n)$ [Varadarajan 1998]
- Gen. Pos. Bichromatic: $O(n^2 \text{polylog}\, n)$ [Kaplan et al. 2020]
- Convex position: $O(n \log n)$ [Marcotte & Suri 1991]

Known Results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position			$\mathcal{NP}$-hard	
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position			$\mathcal{NP}$-hard	
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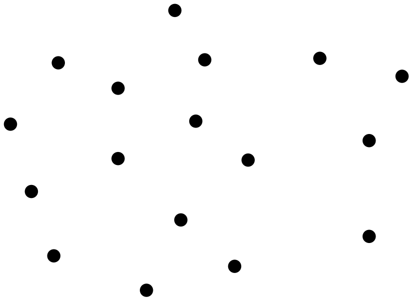
Points in General position



Monochromatic points: Feasibility criterion

Definition

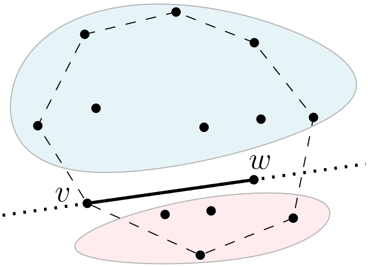
(v, w) is feasible if it is contained in a matching.

Lemma

(v, w) is infeasible iff

- *v and w on the convex hull* CH,
- *odd number of points on each side of the connecting line.*

feasible



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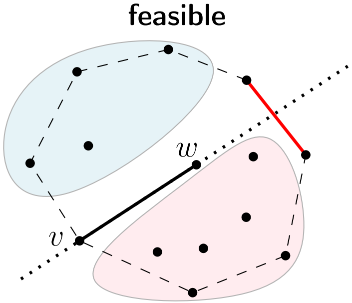

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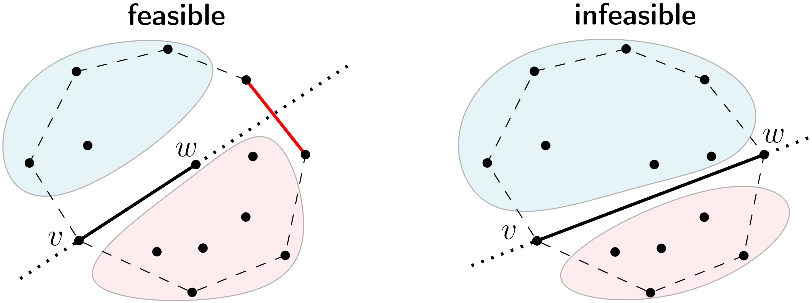
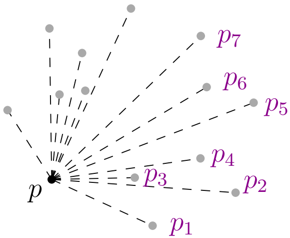


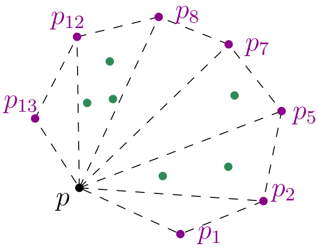
Weak Radial Orderings

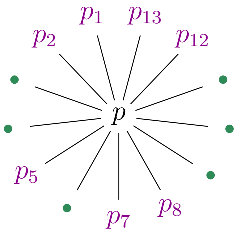
- Radial ordering of p: sorted order of $P \setminus p$ by angle around p



Weak Radial Orderings

- Radial ordering of p: sorted order of $P \setminus p$ by angle around p
- Weak radial ordering of p: radial ordering of p with points of $P \setminus \mathrm{CH}$ being indistinguishable





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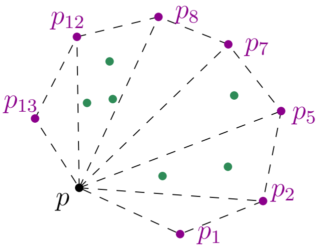
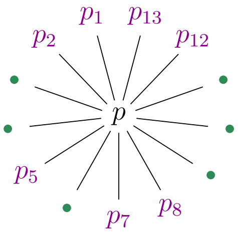




Computation

- Radial orderings of all points in $O(n^2)$ (standard algorithm using dual line arrangement)
- Weak radial orderings of all points in CH in $O(nh)$, $h = |\mathrm{CH}|$

Monochromatic MinMin & MaxMax

Algorithm:

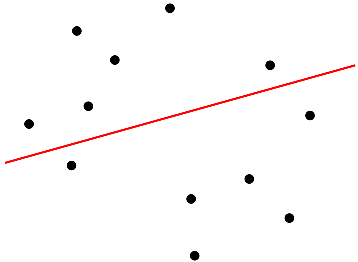
Find shortest/longest feasible edge in 2 steps

- Edges with at least one point in $P \setminus \mathrm{CH}$ in $O(n \log n)$
  (using a Voronoi diagram)
- Edges with both points in CH in $O(nh)$
  (using the weak radial orderings)

Total running time: $O(nh + n \log n)$

Monochromatic MinMin &MaxMax (alternative approach)

Halfplane range queries



How many points on each side of a line?

Monochromatic MinMin &MaxMax (alternative approach)

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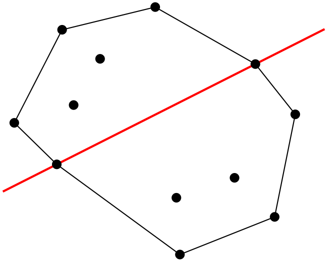


How many points on each side of a line?

Applying [Matoušek 1993] yields algorithm with running time $O(n^{1+\epsilon} + n^{2/3} h^{4/3} \log^3 n)$.

[Opposed to the previous $O(nh + n \log n)$.]

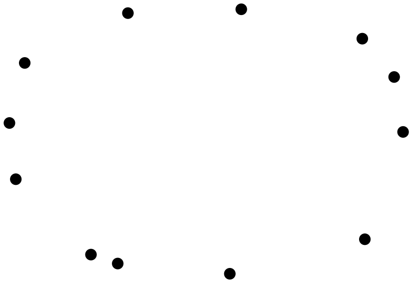
Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$, $O(n^{1+\epsilon} + n^{2/3} h^{4/3} \log^3 n)$		$\mathcal{NP}$-hard	?
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	

Open problems:

- Bichromatic: Edge feasibility criterion checked in polynomial time?
- Is MaxMin $\mathcal{NP}$-hard?

Points in Convex position



Points in Convex position

Classic dynamic programming approach works for all variants:

- $O(n^3)$ time and $O(n^2)$ space complexity.
- Used for MinMax [Biniaz et al. 2014, Carlsson et al. 2015]

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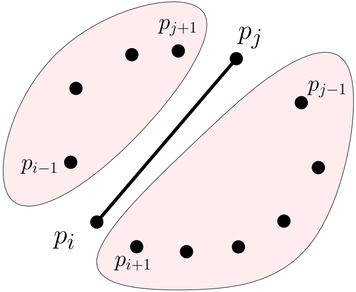


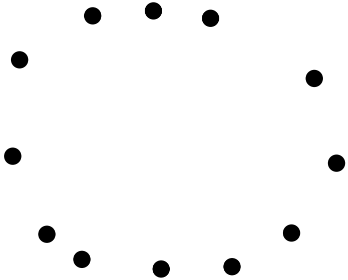
- A subproblem is defined by a sequence of consecutive points.
- For each (sub)problem we test all possible splits into two of its subproblems.

Points in Convex position

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n\log n)$, $O(n^{1+\epsilon} + n^{2/3}h^{4/3}\log^3 n)$		$\mathcal{NP}$-hard	?
Convex Position	$O(n^3)$	$O(n^3)$	$O(n^2)$	$O(n^3)$
Points on circle				
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n^3)$	$O(n^3)$	$O(n^2)$	$O(n^3)$
Points on circle				

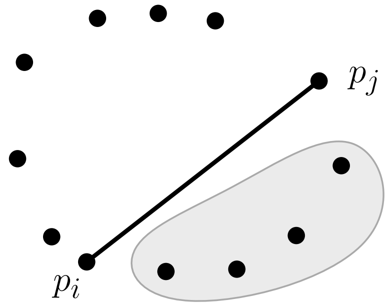
- We can do better for some cases.

Convex Monochromatic MinMin & MaxMax



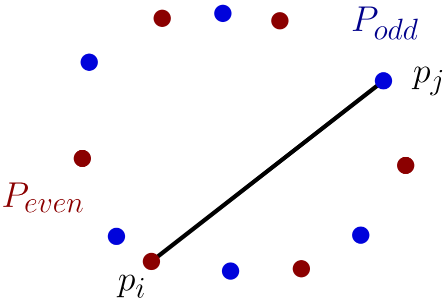
Convex Monochromatic MinMin & MaxMax

- Edge (p_i, p_j) is feasible iff $j - i$ is odd.



Convex Monochromatic MinMin & MaxMax

- Edge (p_i, p_j) is feasible iff $j - i$ is odd.
- Partition P in P_{even} and P_{odd}.



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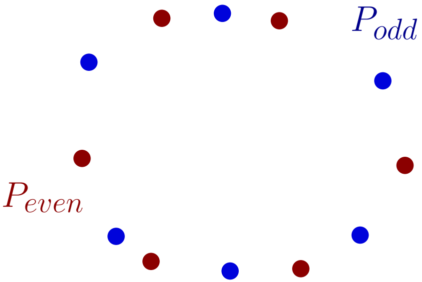
- Edge (p_i, p_j) is feasible iff $j - i$ is odd.
- Partition P in P_{even} and P_{odd}.
- Goal: Find (p_i, p_j) with $p_i \in P_{even}, p_j \in P_{odd}$ of
  - minimum length.



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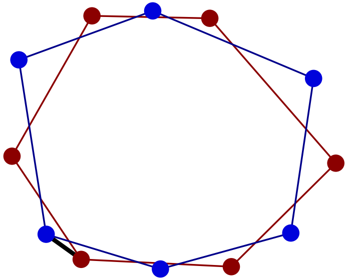

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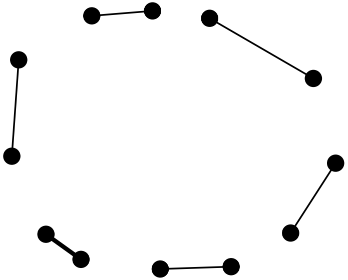
- Match remaining points using boundary edges.



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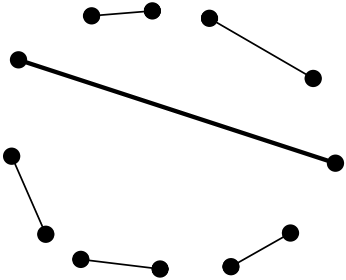
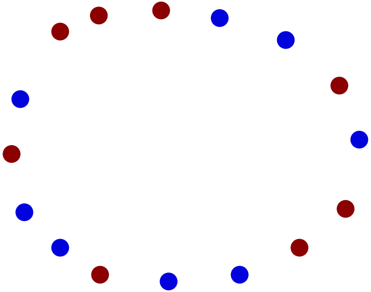


Convex Bichromatic MinMin & MaxMax



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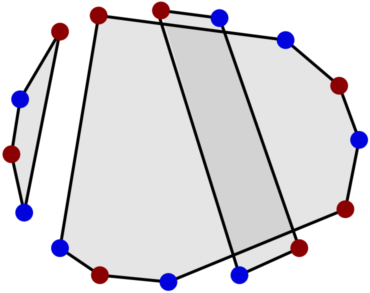
- Partition P into *orbits* [Savić & Stojaković 2018]
  - Takes $O(n)$ time.



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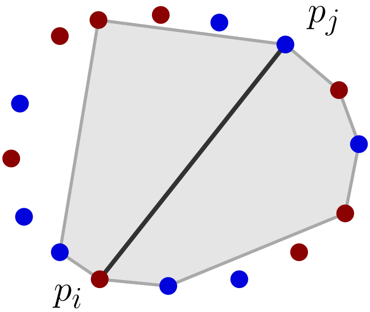

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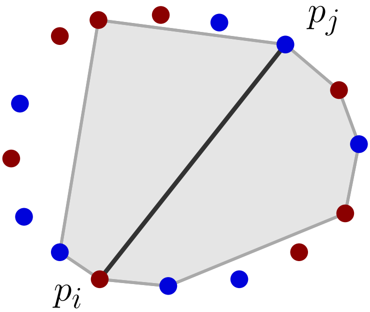

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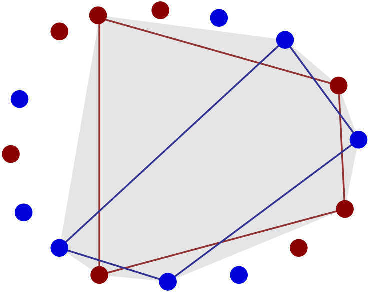

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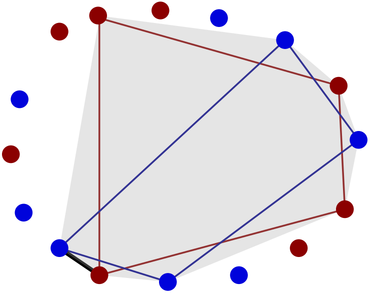

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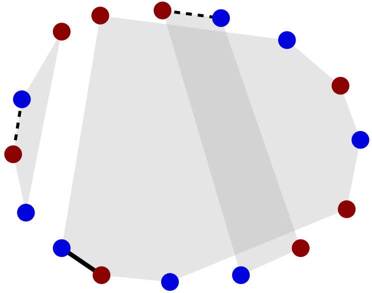

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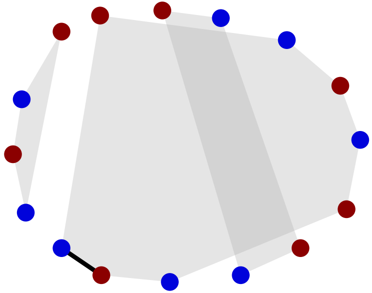

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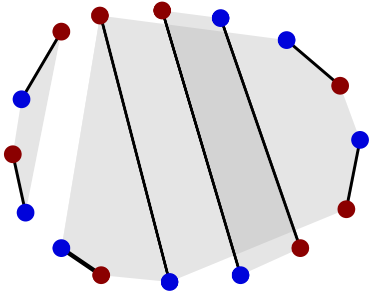
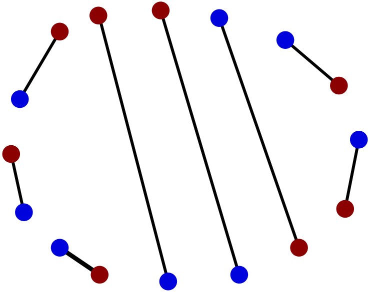


Convex Bichromatic MinMin & MaxMax

- Partition P into *orbits* [Savić & Stojaković 2018]
  - Takes $O(n)$ time.
  - Edge (p_i, p_j) is feasible iff p_i and p_j are in the same orbit.
  - Colors inside an orbit alternate.
- Apply the monochromatic approach to each orbit.
- $O(n)$ time total.



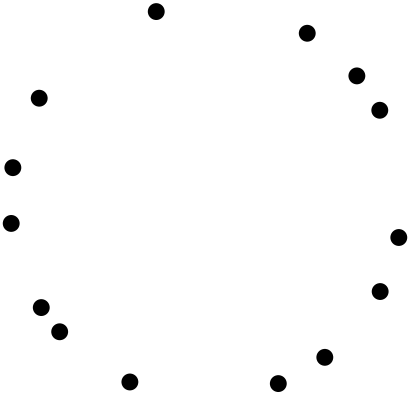
Summary of results

Monochromatic	MinMin	MaxMax	MinMax	MaxMin
General Position	$O(nh + n\log n)$, $O(n^{1+\epsilon} + n^{2/3}h^{4/3}\log^3 n)$		$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$

Open problem:

- Design $o(n^3)$ time algorithms for MaxMin.

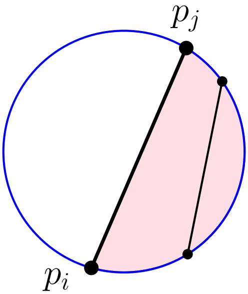
Points on a Circle



Points on Circle

Decreasing chords property

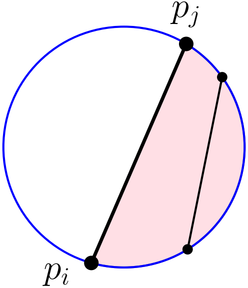
Given an edge (p_i, p_j) all points on (at least) one of its sides have pairiwise distances smaller than p_i and p_j.



Points on Circle

Decreasing chords property

Given an edge (p_i, p_j) all points on (at least) one of its sides have pairiwise distances smaller than p_i and p_j.

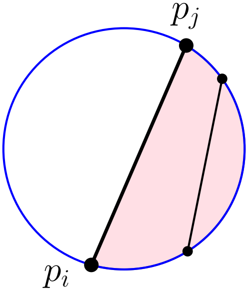


- The shortest edge of a perfect matching is a boundary edge.

Points on Circle

Decreasing chords property

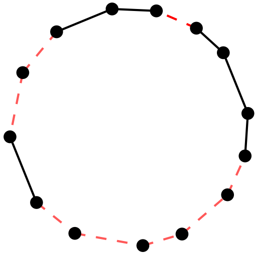
Given an edge (p_i, p_j) all points on (at least) one of its sides have pairiwise distances smaller than p_i and p_j.



- The shortest edge of a perfect matching is a boundary edge.
- Also applies to convex point sets with the decreasing chords property.

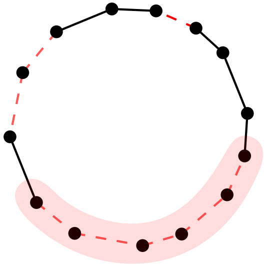
Points on Circle: Monochromatic MaxMin

Idea: Try forbidding all boundary edges shorter than μ, and check if there is still a matching.



Points on Circle: Monochromatic MaxMin

Idea: Try forbidding all boundary edges shorter than μ, and check if there is still a matching.



Lemma

There exists a matching without the forbidden edges iff the length of the longest forbidden chain is less than $n/2$.

Points on Circle: Monochromatic MaxMin

Problem: Find maximal μ such that there is a matching.

- Idea 1: Binary search - $O(n \log n)$ time

Points on Circle: Monochromatic MaxMin

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- Idea 2:

$$\mu_{max} = \min_{i \in \{0,\ldots,n-1\}} \max_{j \in \{i,\ldots,i+\frac{n}{2}-1\}} |p_j p_{j+1}|$$

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  - Use RMQ data structure or sliding window minimum algorithm - $O(n)$ time.

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  - Use RMQ data structure or sliding window minimum algorithm - $O(n)$ time.
  - It is also possible to construct the optimal solution in $O(n)$ time.

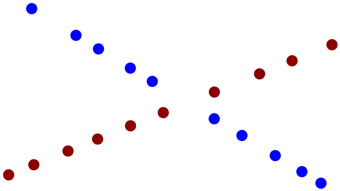
Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$, $O(n^{1+\epsilon} + n^{2/3}h^{4/3}\log^3 n)$		$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$

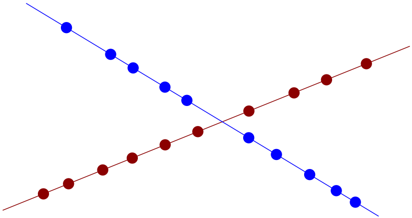
Open problem:

- Bichromatic MaxMin circle in $o(n^3)$ time? Possibly employing the theory of orbits?

Doubly collinear points



Doubly collinear points



Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n\log n)$, $O(n^{1+\epsilon} + n^{2/3}h^{4/3}\log^3 n)$		$\mathcal{NP}$-hard	?
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Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$
Doubly collinear	$O(n)$	$O(n)$	$O(n^4 \log n)$	?

Open problem:

- Is the MaxMin doubly collinear problem $\mathcal{NP}$-hard?

Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$	$O(nh)$	$\mathcal{NP}$-hard	?
	$O(n^{1+\epsilon} + n^{2/3} h^{4/3} \log^3 n)$			
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$
Doubly collinear	$O(n)$	$O(1)$	$O(n^4 \log n)$	?

Thank you for your attention!