The Voronoi Diagram of Rotating Rays with applications to Floodlight Illumination

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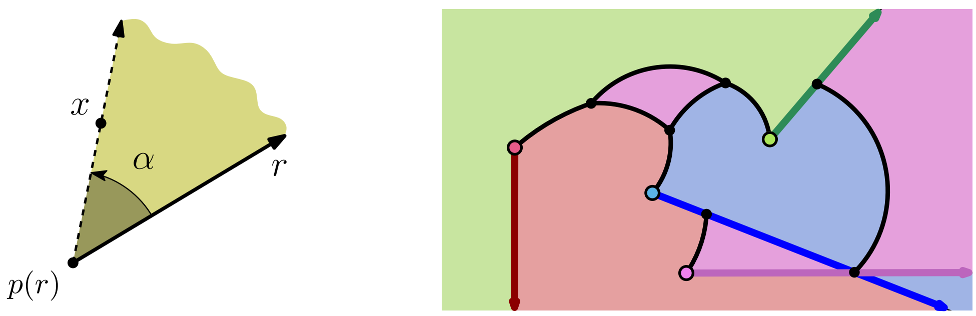
Abstract

We introduce the *Rotating Rays Voronoi diagram*, a Voronoi structure where the input sites are rays and the distance function is the counterclockwise angular distance. This novel diagram can be used to solve illumination or coverage problems where a domain has to be covered by floodlights/wedges of uniform angle. We present structural properties, combinatorial complexity bounds, and algorithms to construct the diagram. Moreover, we show how we can use this Voronoi diagram to compute the *Brocard angle* of a convex polygon in optimal linear time.

1 Introduction

In this work, we study a Voronoi diagram where the input is a set of n rays in the plane, and the distance from a point x to a ray r is the angular distance, i.e., the minimum angle α such that, after counterclockwise rotating r around its apex by α, r *illuminates* x; see Fig. 1.



(a) **(b)**

Figure 1 (a) The angular distance α from a point x to a ray r. (b) The Voronoi diagram of four rays in the plane. All points in a region are first illuminated by the ray with the respective color.

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This is an extended abstract of a presentation given at EuroCG'21. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

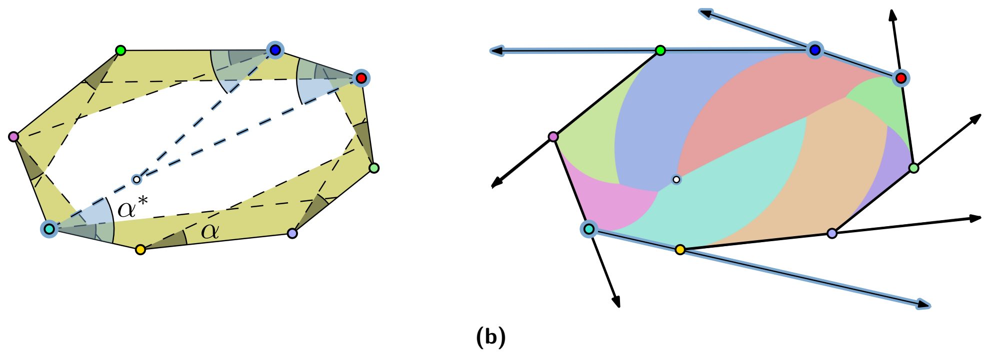


Figure 2 (a) Illumination with edge-aligned α-floodlights. (b) The Rotating Rays Voronoi diagram. Highlighted are the point and the three rays that realize the Brocard angle α^*.

Motivation. The distance measure is motivated by the following illumination problem: An α-*floodlight* is a light source that illuminates a cone with aperture α from its apex. Given a simple polygon P, an α-floodlight is placed on each vertex $v \in P$ facing the interior of P in such a way, that one of its rays contains the successor of v in the counterclockwise order of the vertices of P; see Fig. 2a. The *Brocard Illumination problem* [1] asks for the *Brocard angle*, the smallest value of α for which a set of α-floodlights covers the interior of P.

Constructing the Voronoi diagram of rays inside a convex polygon reveals the Brocard angle, as the latter is realized at a vertex of the diagram with maximum distance; see Fig. 2b. Interestingly, given a set of rays, similar illumination problems can be defined in different domains. The domain may be the entire plane or even a curve and, analogously, constructing the Voronoi diagram in that domain yields the Brocard angle. Hence, there is an interest in studying such diagrams, and designing construction algorithms for different domains.

Our contribution. We introduce the *Rotating Rays Voronoi Diagram* and prove a series of results, paving the way for future work on similar problems. We consider the diagram in the plane and identify structural properties which we complement with combinatorial complexity results, a worst case $\Omega(n^2)$ lower bound and an $O(n^{2+\epsilon})$ upper bound, together with an $O(n^{2+\epsilon})$-time construction algorithm. Finally, motivated by applications to illumination problems, we restrict our domain to a convex polygonal region bounded by the input set of sites, and construct the Voronoi diagram in such a domain in optimal $\Theta(n)$ time.

Related work. In the Brocard illumination problem, a polygon is called a *Brocard polygon* [2] if all the α-floodlights simultaneously illuminate a point inside the polygon when α is equal to the Brocard angle. The characterization of Brocard polygons has a long history, yet, only harmonic polygons (which include triangles and regular polygons) are known to be Brocard [5]. Deciding whether a polygon is Brocard can be done in $O(n)$ time and then the Brocard angle can be computed in $O(1)$ time. Computing the Brocard angle of simple polygons was first studied by Alegría et al. [1]. The authors solved the problem in $O(n^3 \log^2 n)$ time, and complemented this result with an $O(n \log n)$-time algorithm for convex polygons.

Since their introduction [4], floodlight illumination problems have been widely studied, see e.g. [15, 20]. The case when the floodlights are of uniform angle is of particular interest, and has been explored by several authors, see e.g. [6, 9, 10, 16, 19]. Rotating α-floodlights are also used to model devices with limited sensing range, like surveillance cameras or directional antennae [7, 13, 14]. The Rotating Rays Voronoi diagram seems to be novel with respect to both the input sites and the distance function. A related diagram was defined [8] to study dominance regions of players in the analysis of football (soccer) matches [18].

2 Preliminaries

In this section we formally define the Rotating Rays Voronoi diagram. Let $\mathcal{S}$ be a set of n rays in the plane. Given a ray r, we denote its apex by $p(r)$ and its supporting line by $l(r)$.

▶ **Definition 2.1.** Given a ray r and a point $x \in \mathbb{R}^2$, the *angular distance* from x to r, denoted $d_\angle(x, r)$, is the minimum counterclockwise angle α from r to a ray with apex on $p(r)$ passing through x; see Fig. 1a.

▶ **Definition 2.2.** Given two rays r and s, the *dominance region* $dr(r, s)$ is the set of points with smaller angular distance to r than to s, i.e., $dr(r, s) = \{x \in \mathbb{R}^2 \mid d_\angle(x, r) < d_\angle(x, s)\}$. The *angular bisector* of r and s, denoted $b_\angle(r, s)$, is the curve delimiting $dr(r, s)$ and $dr(s, r)$.

The different types of bisectors are illustrated in Fig. 3. Given two rays r and s, let $I = l(r) \cap l(s)$. The bisector $b_\angle(r, s)$ is the union of r, s, and a circular arc a that connects $p(r)$ to $p(s)$. The arc a belongs to the *bisecting circle* $C_b(r, s)$, which we define as follows:

- If I, $p(r)$, and $p(s)$ are pairwise different, then $C_b(r, s)$ is the circle through I, $p(r)$, and $p(s)$. The arc a contains I if, and only if, I lies either on none or on both of r and s.
- If $I = p(r)$ and $I \neq p(s)$, then $C_b(r, s)$ is the circle tangent to $l(r)$ passing through $p(r)$ and $p(s)$. Both a and r lie on the same side of $l(s)$ if, and only if, $p(r)$ lies on s. We analogously define $C_b(r, s)$ if $I = p(s)$ and $I \neq p(r)$.
- If $p(r) = p(s)$, then both $C_b(r, s)$ and a degenerate to a single point.
- If $l(r)$ and $l(s)$ are parallel, then $C_b(r, s)$ degenerates to the line through $p(r)$ and $p(s)$, and a degenerates either to a line segment or to two halflines.

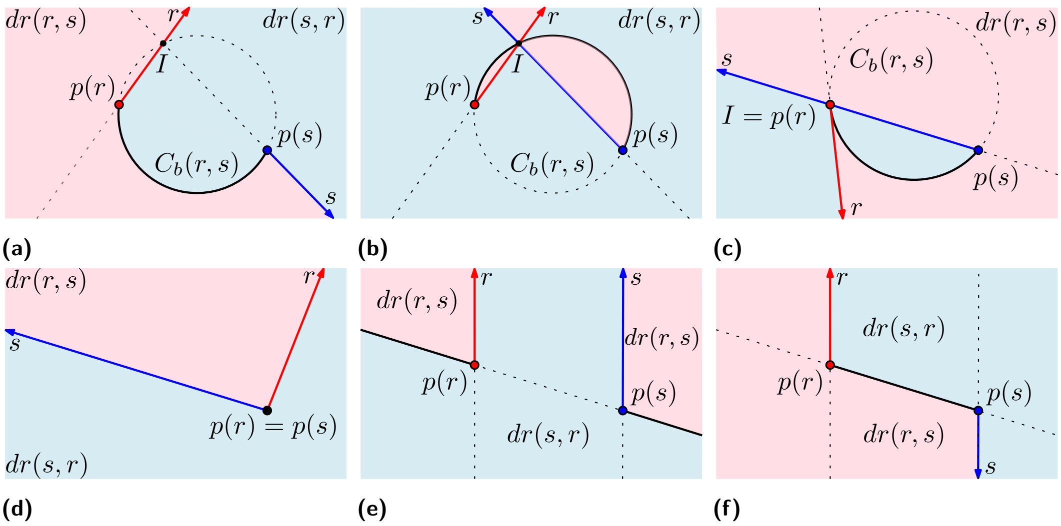


Figure 3 The bisector of two rays r and s which are: (a) non-intersecting, (b) intersecting, (c) with $p(r)$ lying on s, (d) sharing their apex, (e) parallel, and (f) anti-parallel.

▶ **Definition 2.3.** The *Rotating Rays Voronoi diagram* of a set $\mathcal{S}$ of rays is the subdivision of $\mathbb{R}^2$ into *nearest Voronoi regions* defined as follows:

$$vreg(r) := \{\, x \in \mathbb{R}^2 \mid \forall s \in \mathcal{S} \setminus \{r\} : \; d_\angle(x, r) < d_\angle(x, s) \,\}.$$

The graph structure of the diagram of $\mathcal{S}$ is denoted by $\mathrm{RVD}(\mathcal{S}) := \big(\mathbb{R}^2 \setminus \bigcup_{r \in \mathcal{S}} vreg(r)\big) \cup \mathcal{S}$.

An example diagram is shown in Fig. 1b. A region $vreg(r)$ can be equivalently defined as the intersection of all the dominance regions of r, i.e., $vreg(r) = \bigcap_{s \in \mathcal{S} \setminus \{r\}} dr(r, s)$. Note that a region can have more than one connected component, which we call a *face* of the region.

3 RVD: properties, complexity, and an algorithm

In this section we study the diagram in the plane. Assuming that no two rays of $\mathcal{S}$ are parallel to each other, the following two structural properties hold.

▶ **Lemma 3.1.** *RVD($\mathcal{S}$) has exactly n unbounded faces, one for each ray.*

Proof sketch. Let Γ be a circle containing all the bisecting circles of $\mathcal{S}$. The intersection of Γ and the unbounded part of each dominance region of $r \in \mathcal{S}$ is a circular arc with endpoint $r \cap \Gamma$. The intersection of all such arcs is connected. Hence $vreg(r) \cap \Gamma$ is connected. ◀

▶ **Lemma 3.2.** *RVD($\mathcal{S}$) is connected.*

Proof sketch. If RVD($\mathcal{S}$) is not connected, then there is a region $vreg(r)$ that either has an unbounded face with two occurrences at infinity or it encloses a component of RVD($\mathcal{S}$), creating an *"island"*. The first case is excluded by Lemma 3. The second case implies that such an island also exists in some dominance region of the site r, which leads to a contradiction. ◀

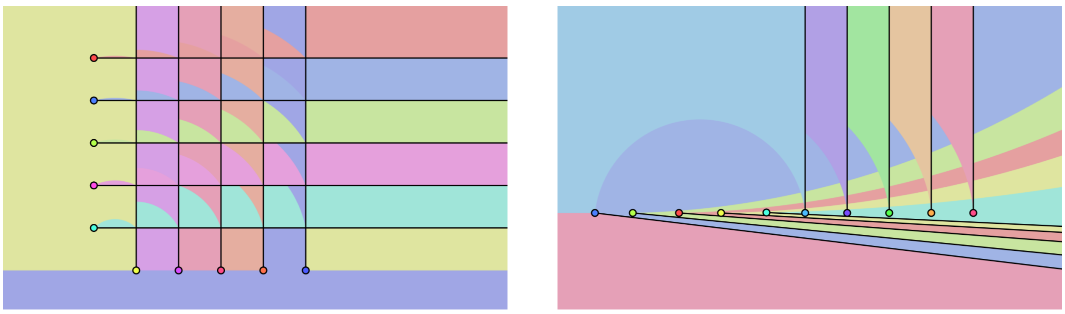
We now show two lower bound constructions for the worst case complexity of the RVD.

▶ **Theorem 3.3.** *RVD($\mathcal{S}$) has $\Omega(n^2)$ combinatorial complexity in the worst case. This holds even if the rays are pairwise non-intersecting.*

Proof sketch. The bound is achieved by creating a grid structure in which the rays have $\Theta(n^2)$ intersections, each inducing a vertex in RVD($\mathcal{S}$); see Figure 4a. A $\Theta(n^2)$ construction can also be achieved by a set of rays that are pairwise disjoint. An example is shown in Figure 4b, where the Voronoi regions of the $n/2-1$ leftmost rays have $n/2+1$ faces each. ◀

▶ **Lemma 3.4.** *A Voronoi region of RVD($\mathcal{S}$) has $\Theta(n^2)$ complexity in the worst case.*

Proof sketch. Any vertex incident to a region $vreg(r)$, is defined by r and a pair of rays. Further, the diagram of three rays has $O(1)$ complexity, so the $O(n^2)$ bound follows. For the lower bound, we create a grid structure as in Theorem 3.3. We then add a ray r so that $vreg(r)$ has a face inside each cell of the grid, thus $vreg(r)$ has $\Theta(n^2)$ complexity; see Fig. 5. ◀



(a) **(b)**

Figure 4 RVD($\mathcal{S}$) with $\Theta(n^2)$ complexity, where the rays in $\mathcal{S}$ are (a) intersecting and (b) pairwise non-intersecting.

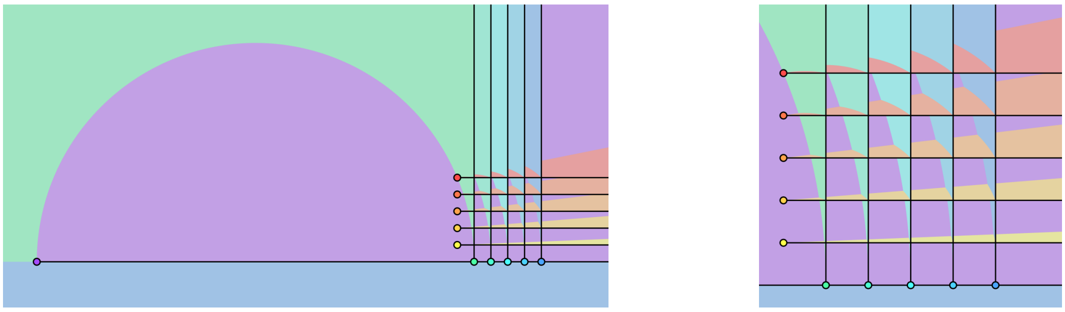


Figure 5 RVD($\mathcal{S}$) with a region having $\Theta(n^2)$ complexity (a) zoomed out and (b) zoomed in.

Finally, we apply the general upper bound by Sharir [17] to yield a near quadratic upper bound on the complexity of the RVD. This is accompanied by a construction algorithm.

▶ **Theorem 3.5.** *For any $\epsilon > 0$, RVD($\mathcal{S}$) has $O(n^{2+\epsilon})$ combinatorial complexity. Further, RVD($\mathcal{S}$) can be constructed in $O(n^{2+\epsilon})$ time.*

Proof sketch. Each site induces a distance function which maps each point in the plane to its distance to that site. The RVD can be seen as a minimization diagram of these distance functions. With a monotone increasing transformation we map these distance functions to algebraic functions. Applying the transformations keeps the lower envelope of the distance functions invariant. The lower envelope of these n algebraic functions has $O(n^{2+\epsilon})$ combinatorial complexity and it can be constructed in $O(n^{2+\epsilon})$ time [17]. ◀

We can extend the Brocard Illumination problem to the plane as follows. We place an α-floodlight f_r on every $r \in \mathcal{S}$ such that f_r is equal to r when $\alpha = 0$. We want the minimum angle α^* for which the set $\{f_r \mid r \in \mathcal{S}\}$ of α^*-floodlights illuminates the plane. The angle α^* is realized at a point of maximum distance, which is either a vertex of RVD($\mathcal{S}$) or a point at infinity on a ray of $\mathcal{S}$. Hence, we can find α^* by constructing RVD($\mathcal{S}$) and then traversing the diagram in linear time in its size. Note that α^* takes values in the interval $(2\pi/n, 2\pi)$.

4 RVD of a convex polygon: Brocard illumination

We now turn our attention to the Brocard illumination problem. Given a convex polygon P with n vertices, we describe an algorithm to compute the Brocard angle α^* of P by means of the RVD. Let $\mathcal{S}_P$ be the set of n rays such that each ray has a vertex $v \in P$ as apex, and passes through the successor of v in the counterclockwise order of the vertices of P. Let PRVD($\mathcal{S}_P$) be the part of RVD($\mathcal{S}_P$) restricted to the interior of P. Note that PRVD($\mathcal{S}_P$) has $\Theta(n)$ complexity, as opposed to RVD($\mathcal{S}_P$) which can have $\Theta(n^2)$. We show the following.

▶ **Theorem 4.1.** *Given a convex polygon P, we can construct PRVD($\mathcal{S}_P$) in $\Theta(n)$ time. The Brocard angle of P can also be found in the same time.*

Algorithm outline. We first partition $\mathcal{S}_P$ into four sets $\mathcal{S}_N, \mathcal{S}_W, \mathcal{S}_S$ and $\mathcal{S}_E$ of consecutive rays, such that any two rays in a subset have an angular difference at most $\pi/2$, see Fig. 6a. For each set $\mathcal{S}_d$, $d \in \{$N,W,S,E$\}$, we obtain a set $\mathcal{S}_d^r$ in which every ray of $\mathcal{S}_d$ is rotated by an angle of $-\pi/2$. Then, we construct each diagram RVD($\mathcal{S}_d^r$) independently, see Fig. 6c. Finally, we merge the four diagrams in two steps to obtain PRVD($\mathcal{S}_P$), see Figs. 7 and 8.

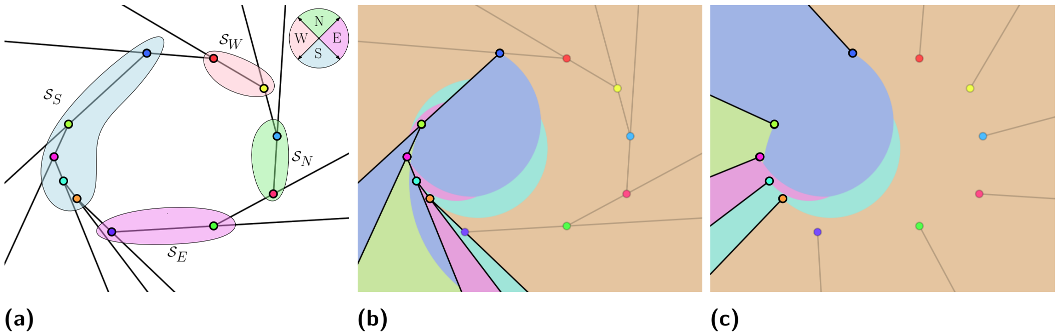


Figure 6 (a) Partition of $\mathcal{S}_P$ into sets $\mathcal{S}_N, \mathcal{S}_W, \mathcal{S}_S$ and $\mathcal{S}_E$. (b) RVD($\mathcal{S}_S$). (c) RVD($\mathcal{S}_S^r$)

Constructing the diagrams. To construct the diagram of each subset $\mathcal{S}_d^r$ in optimal $\Theta(|\mathcal{S}_d^r|)$ time, we make use of the *abstract Voronoi diagrams* framework [11, 12]. To fall under this framework, the underlying system of bisectors must satisfy the following three *axioms*:

A1 The bisector $b_\angle(r,s)$, $\forall r,s \in \mathcal{S}_d^r$, is an unbounded Jordan curve.
A2 The region $vreg(r)$ in RVD($\mathcal{S}'$), $\forall \mathcal{S}' \subseteq \mathcal{S}_d^r$ and $\forall r \in \mathcal{S}'$, is connected.
A3 The closure of the union of all regions in RVD($\mathcal{S}'$), $\forall \mathcal{S}' \subseteq \mathcal{S}_d^r$, covers $\mathbb{R}^2$.

▶ **Lemma 4.2.** *The system of bisectors of $\mathcal{S}_d^r$ satisfies the axioms A1-A3.*

Proof sketch. A1 is satisfied since the rays are disjoint. A2 holds due to the rotation of $-\pi/2$: after the rotation, no ray intersects twice any bisecting circle, thus no bounded faces appear, see e.g., Figs. 6b and 6c. A3 is satisfied as each ray induces a distance function. ◀

The intuition for the rotation is twofold: On PRVD($\mathcal{S}_P$) only circular parts of bisectors appear and under $-\pi/2$ rotation the circular parts of the bisectors remain the same. Note that there does not exist a rotation angle for which the complete set $\mathcal{S}_P$ satisfies axiom A2.

To construct RVD($\mathcal{S}_d^r$) we use the $\Theta(n)$-time algorithm of [3]. Apart from satisfying axioms A1-A3, [3] requires that the order of the regions of RVD($\mathcal{S}'$) along a simple curve is known, for any $\mathcal{S}' \subseteq \mathcal{S}_d^r$. Since, the rays are non-intersecting, this order coincides with the order of the rays of $\mathcal{S}'$ along the boundary of the polygon. We obtain the following lemma.

▶ **Lemma 4.3.** *RVD($\mathcal{S}_d^r$) is a tree of $\Theta(|\mathcal{S}_d^r|)$ complexity. Further, RVD($\mathcal{S}_d^r$) can be constructed in $\Theta(|\mathcal{S}_d^r|)$ time.*

Merging the diagrams. We merge all four diagrams, in two steps, to obtain PRVD($\mathcal{S}_P$). In a first step we merge RVD($\mathcal{S}_W^r$) with RVD($\mathcal{S}_S^r$) to obtain RVD($\mathcal{S}_W^r \cup \mathcal{S}_S^r$), where the *merge curve* consists of the first ray in each of $\mathcal{S}_W^r$ and $\mathcal{S}_S^r$ and the set of circular edges, which are equidistant to both sets; see Fig. 7. We respectively obtain RVD($\mathcal{S}_E^r \cup \mathcal{S}_N^r$). Then, in a second step we merge the diagrams RVD($\mathcal{S}_W^r \cup \mathcal{S}_S^r$) and RVD($\mathcal{S}_E^r \cup \mathcal{S}_N^r$), restricted to the interior of P, to obtain PRVD($\mathcal{S}_P$); see Fig. 8. Using standard tracing techniques for Voronoi diagrams and the properties of the angular bisectors, we can prove the following result.

▶ **Lemma 4.4.** *Given RVD($\mathcal{S}_d^r$) $\forall d \in$ {N,W,S,E}, PRVD($\mathcal{S}_P$) can be constructed in $\Theta(n)$ time.*

Concluding this section, the Brocard angle of P, denoted by α^*, is realized at a point of maximum distance in the interior of P which lies on a vertex of PRVD($\mathcal{S}_P$). Hence, we can solve the Brocard Illumination problem in $\Theta(n)$ time, by constructing PRVD($\mathcal{S}_P$) and then traversing it, to obtain α^*. Note that α^* takes values in the interval $(0, \pi/2 - \pi/n]$.

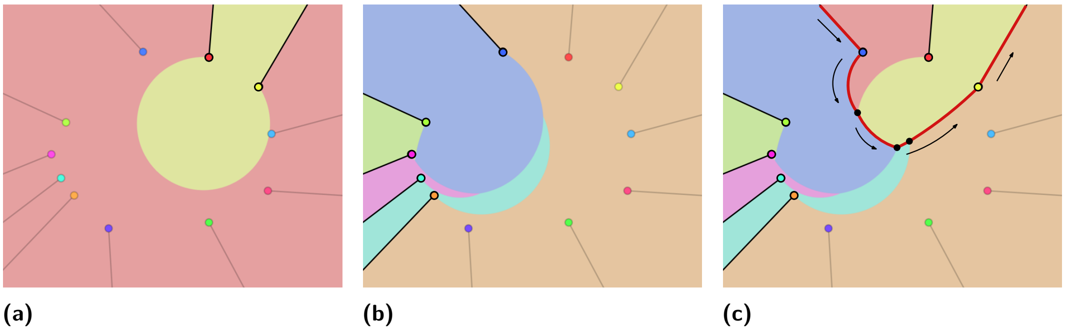


Figure 7 Merging diagrams (a) $\mathrm{RVD}(\mathcal{S}^r_W)$ and (b) $\mathrm{RVD}(\mathcal{S}^r_S)$ into (c) $\mathrm{RVD}(\mathcal{S}^r_W \cup \mathcal{S}^r_S)$.

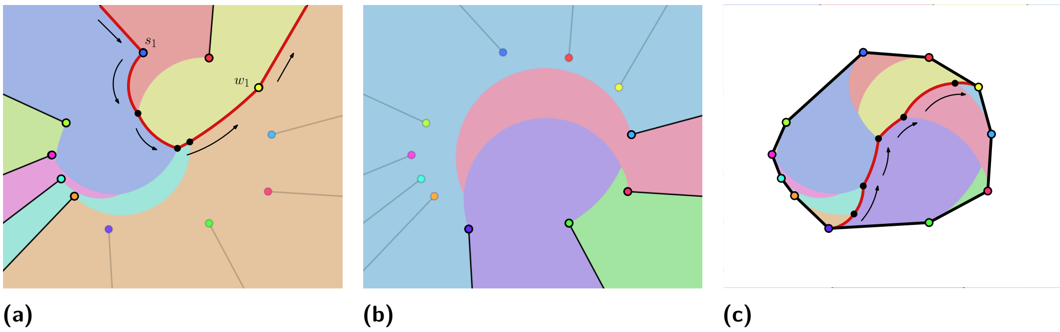


Figure 8 Merging diagrams (a) $\mathrm{RVD}(\mathcal{S}^r_W \cup \mathcal{S}^r_S)$ and (b) $\mathrm{RVD}(\mathcal{S}^r_E \cup \mathcal{S}^r_N)$ into (c) $\mathrm{PRVD}(\mathcal{S}_P)$.

5 Concluding remarks

By means of the Rotating Rays Voronoi diagram, we showed how to find in optimal time the Brocard angle of a convex polygon, settling an interesting geometric problem. Our method is more general: given any domain D and a set of rays $\mathcal{S}$, we can find the minimum angle needed to illuminate D with floodlights aligned at $\mathcal{S}$, by constructing $\mathrm{RVD}(\mathcal{S})$ inside D.

There are many questions to investigate. What is the worst case complexity of the diagram in the plane? Is it $\Theta(n^2)$? Can the diagram in the plane be constructed in time $o(n^{2+\epsilon})$? How does our approach to compute the Brocard angle extend to other classes of polygons?

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