The Voronoi Diagram of Rotating Rays with applications to Floodlight Illumination

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EuroCG 2021
Saint Petersburg, Russia

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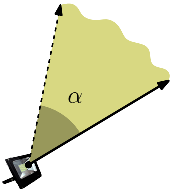
Brocard Illumination

α-floodlight



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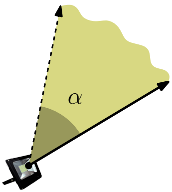

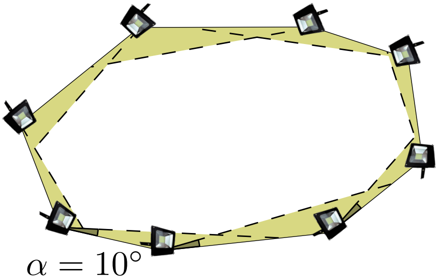

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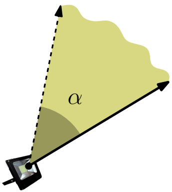

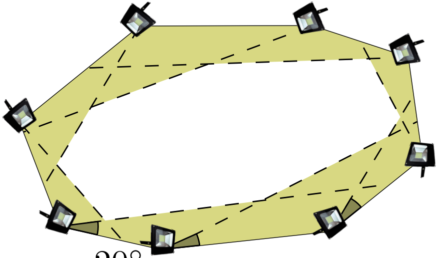

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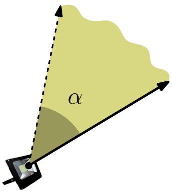

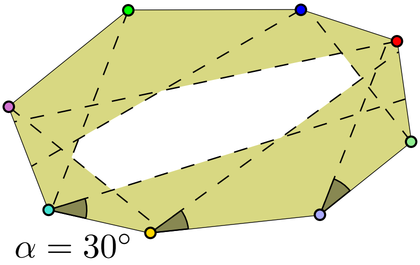

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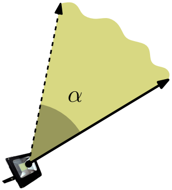

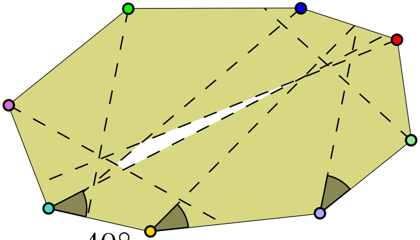

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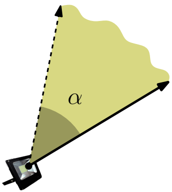

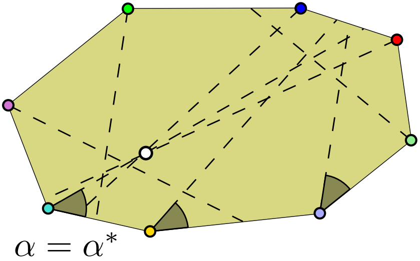

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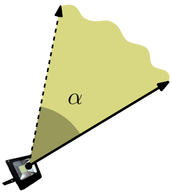
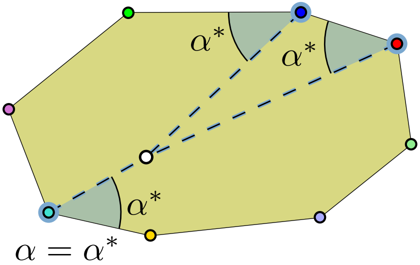
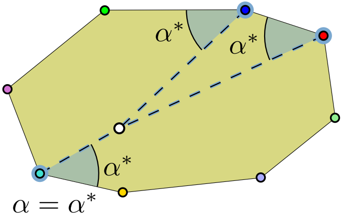
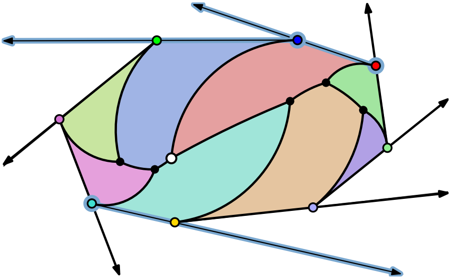




Brocard Illumination

Input: Polygon P with edge-aligned α-floodlights.
Goal: Minimum angle α^* to illuminate P.
Approach: Define the Voronoi Diagram of Rotating Rays.





Related Work

Brocard Polygons - Illumination

- Brocard Polygons (only harmonic polygons)
  e.g. [Casey 1888, Dmitriev & Dynkin 1946, Bernhart 1959]

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- Several variants/results, e.g. [Bose et al. 1993, Uruttia 2000]
- Uniform angle, e.g. [O'Rourke 1995, Toth 2002]

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Application - Domain coverage

- Directional Antennas or Surveillance Cameras [Berman et al. 2007, Kranakis et al. 2011, Neishaboori et al. 2014 ,Czyzowicz et al. 2015]

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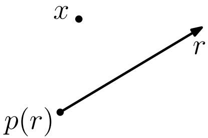
Angular distance - bisectors

Definition

Given a ray r and a point $x \in \mathbb{R}^2$, the angular distance from x to r, $d_\angle(x, r)$,



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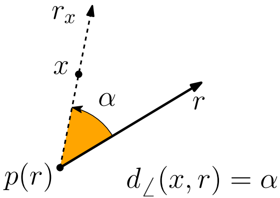
Given a ray r and a point $x \in \mathbb{R}^2$, the angular distance from x to r, $d_\angle(x, r)$, is the minimum counterclockwise angle α from r to a ray with apex $p(r)$ passing through x.



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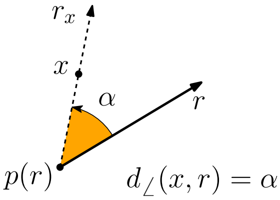
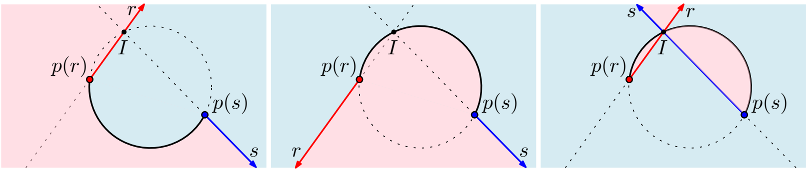


Definition

Given two rays r and s, their angular bisector, $b_\angle(r, s)$, is the curve delimiting the points closer to r and the points closer to s.



Rotating Rays Voronoi Diagram (RVD)

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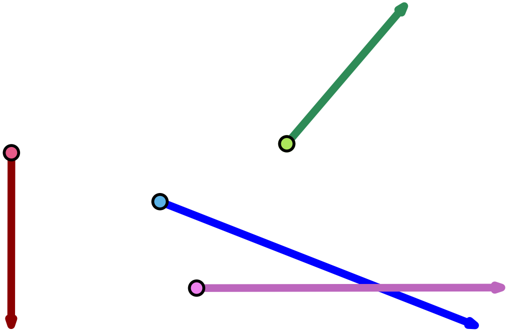
Given a set of rays $\mathcal{S}$.



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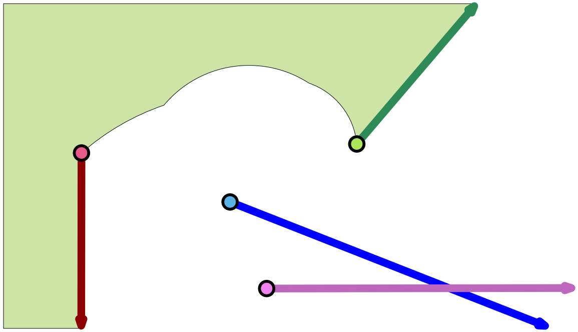
Given a set of rays $\mathcal{S}$. The Voronoi region of a ray $r \in \mathcal{S}$ is:

$$vreg(r) := \{\, x \in \mathbb{R}^2 \mid \forall s \in \mathcal{S} \setminus \{r\} : \; d_\angle(x, r) < d_\angle(x, s) \,\}.$$



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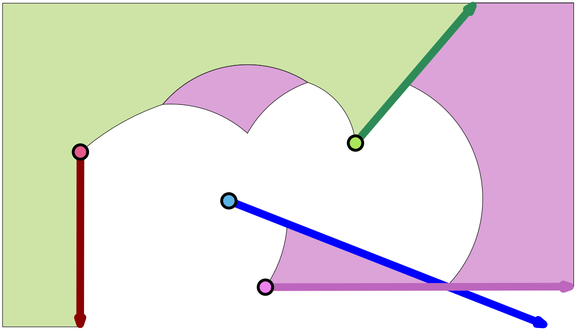

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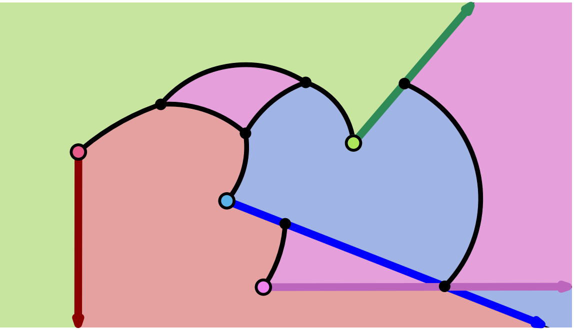
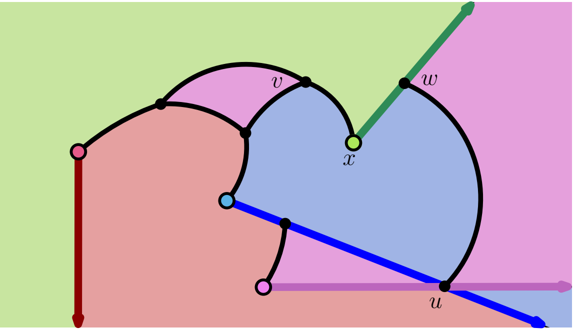
The Rotating Rays Voronoi Diagram of $\mathcal{S}$ is the subdivision of $\mathbb{R}^2$ in Voronoi regions. RVD($\mathcal{S}$) is the graph structure of the diagram.

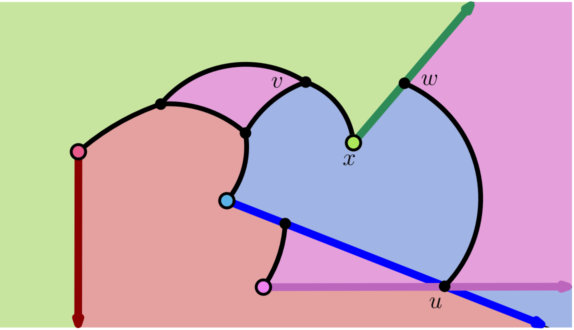


Properties



- RVD($\mathcal{S}$) has different types of vertices and edges.

Properties



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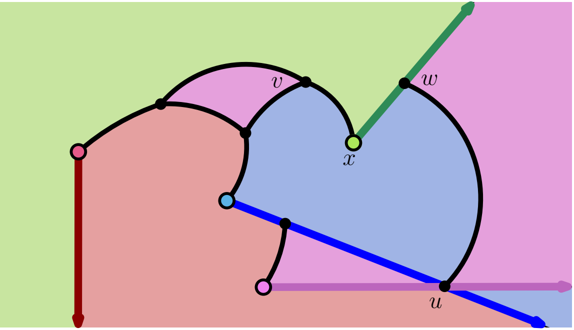

- RVD($\mathcal{S}$) has different types of vertices and edges.
- A region can have many faces; exactly one is unbounded.
- RVD($\mathcal{S}$) is connected.

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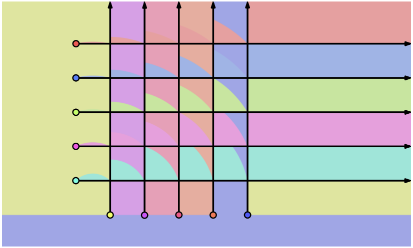
Diagram Complexity: Lower bound

Theorem

Given a set $\mathcal{S}$ of n rays RVD($\mathcal{S}$) *has* $\Omega(n^2)$ *worst case complexity.*



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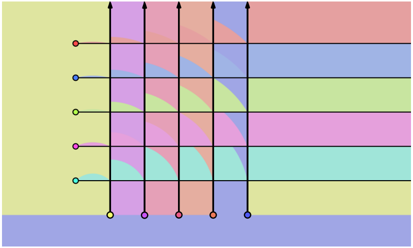

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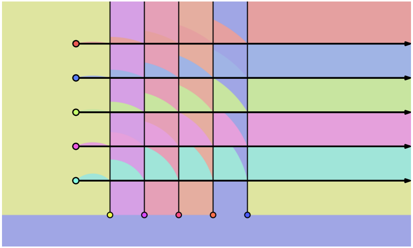

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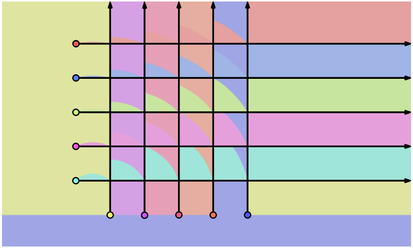

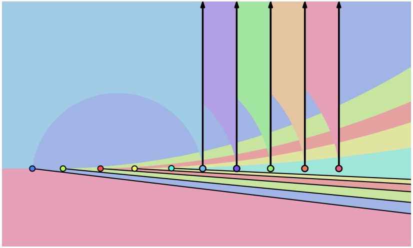

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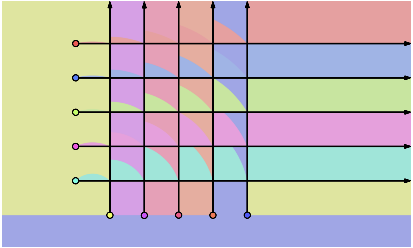

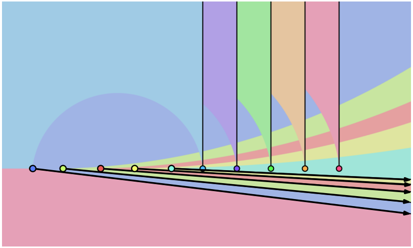

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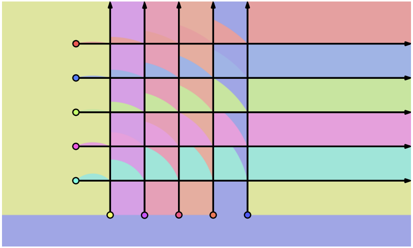

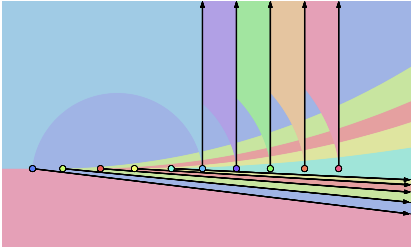

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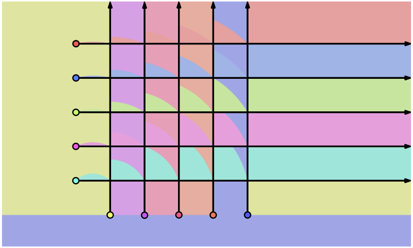

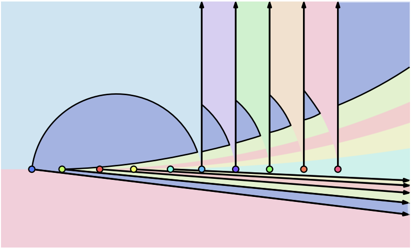

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A region of RVD($\mathcal{S}$) *has* $\Theta(n^2)$ *worst case complexity.*

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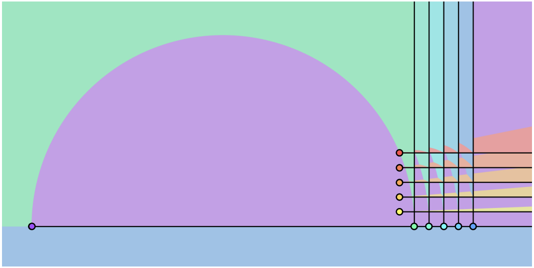

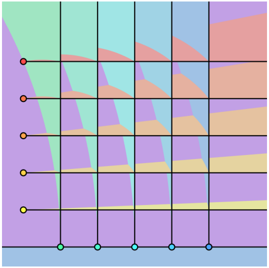

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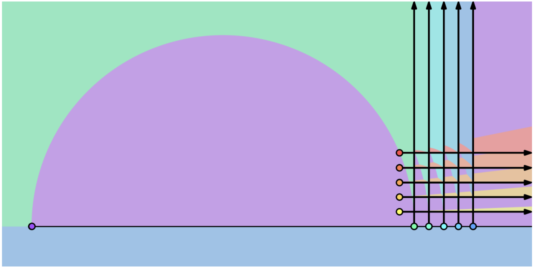

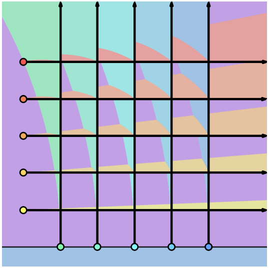

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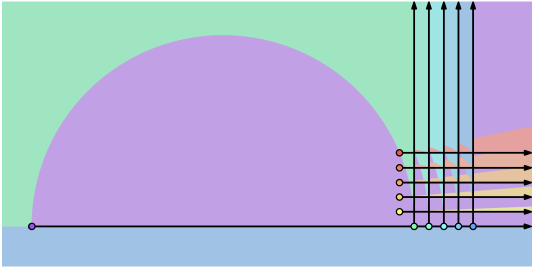

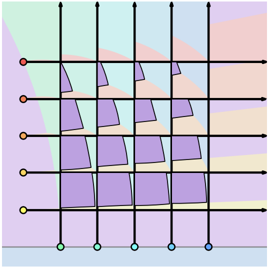

Complexity upper bound & Algorithm

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Given a set $\mathcal{S}$ *of* n *rays* $\mathrm{RVD}(\mathcal{S})$ *has* $O(n^{2+\epsilon})$ *complexity* $\forall \epsilon > 0$.

- Lower envelopes of distance functions in 3-space [Sharir 1994].

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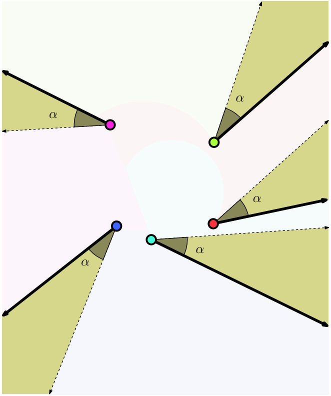

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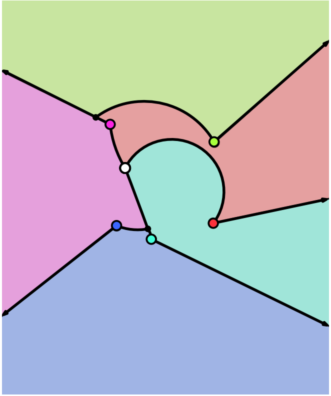

Minimum angle α^* to illuminate $\mathbb{R}^2$

1. Construct RVD($\mathcal{S}$). $O(n^{2+\epsilon})$ time.
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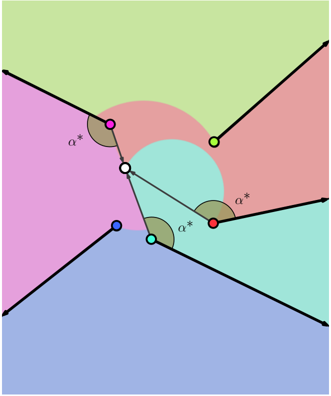

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- $\alpha^* \in (2\pi/n, 2\pi)$

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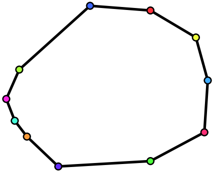

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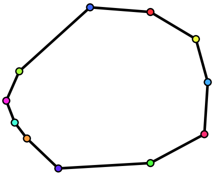

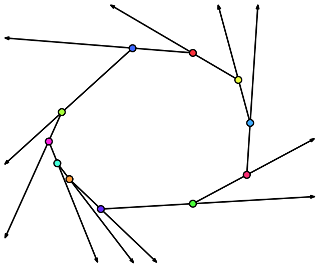

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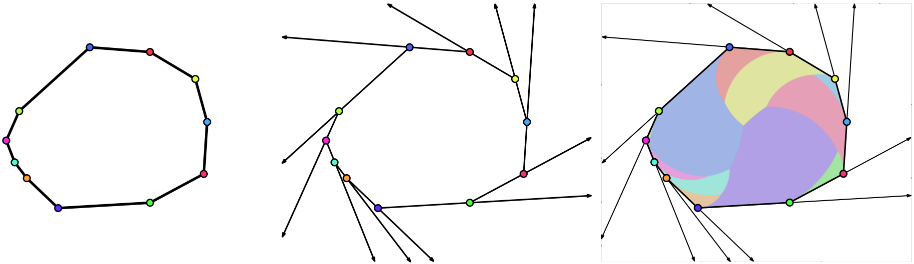
- Output: $\mathrm{PRVD}(\mathcal{S}_P) := \mathrm{RVD}(\mathcal{S}_P) \cap P$.



Brocard Illumination of Polygon

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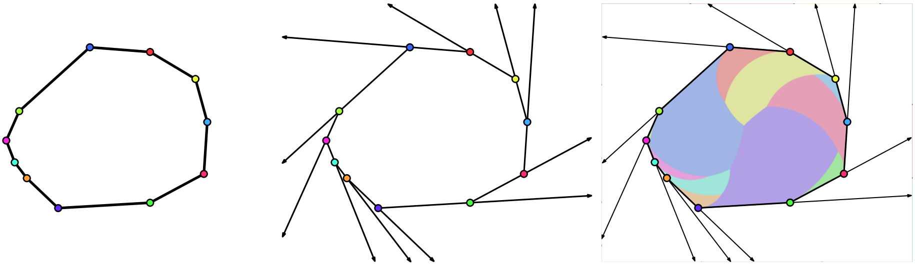
- Output: Brocard angle of P.



Algorithm outline

1. Divide $\mathcal{S}_P$ into 4 sets of rays.
2. Construct the diagrams of the 4 sets.
3. Merge the 4 diagrams to obtain PRVD($\mathcal{S}_P$).

Algorithm outline

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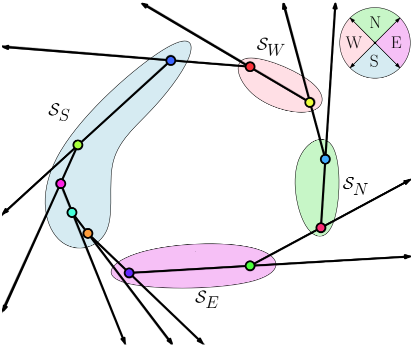
Step 1.

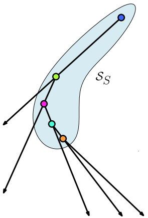
Partition $\mathcal{S}_P$ into 4 sets $\mathcal{S}_N, \mathcal{S}_W$, $\mathcal{S}_S$ and $\mathcal{S}_E$ depending on the direction of the rays.



Constructing the 4 diagrams

Step. 2

For each $\mathcal{S}_d$, $d \in \{$N,W,S,E$\}$:



Use Hamiltonian Abstract Voronoi Diagrams.
[Klein 1989, Klein & Lingas 1994]

- For each $\mathcal{S}' \subseteq \mathcal{S}_d^r$ satisfy a set of axioms.

Constructing the 4 diagrams

Step. 2

For each $\mathcal{S}_d$, $d \in \{$N,W,S,E$\}$: obtain a set $\mathcal{S}_d^r$ in which every ray of $\mathcal{S}_d$ is rotated by $-\pi/2$.

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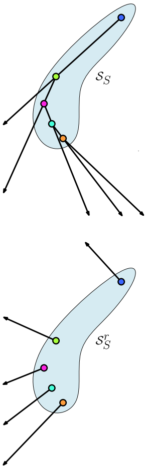

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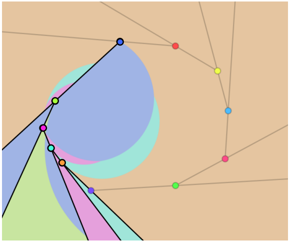
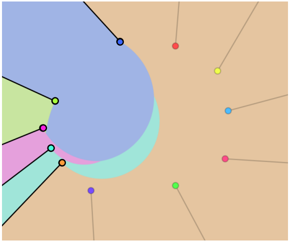




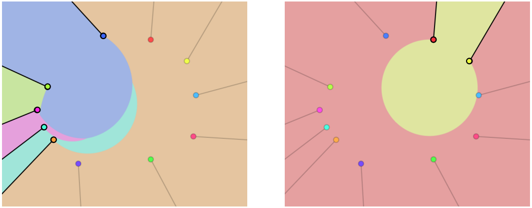
Lemma

RVD($\mathcal{S}_d^r$) *is a tree of* $\Theta(|\mathcal{S}_d^r|)$ *complexity.*
RVD($\mathcal{S}_d^r$) *can be constructed in* $\Theta(|\mathcal{S}_d^r|)$ *time.*

Step 3. Merging the 4 diagrams

Step 3.a.

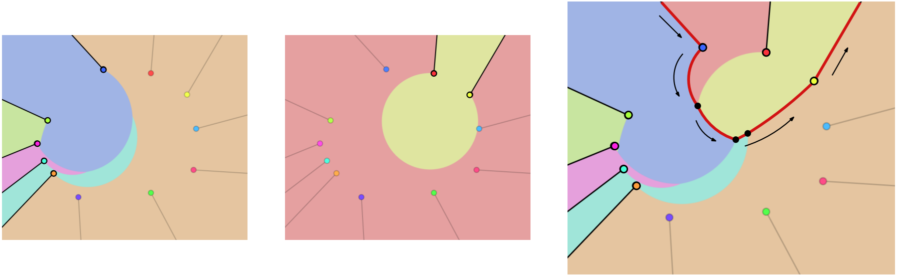
Merge $\mathrm{RVD}(\mathcal{S}^r_W)$ with $\mathrm{RVD}(\mathcal{S}^r_S)$.



Step 3. Merging the 4 diagrams

Step 3.a.

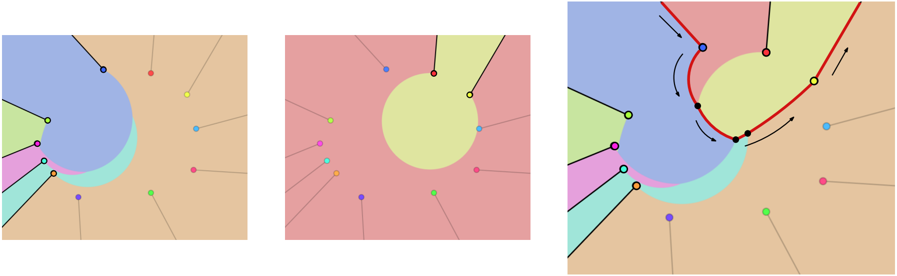
Merge RVD($\mathcal{S}^r_W$) with RVD($\mathcal{S}^r_S$) into RVD($\mathcal{S}^r_W \cup \mathcal{S}^r_S$).



Step 3. Merging the 4 diagrams

Step 3.a.

Merge RVD($\mathcal{S}_W^r$) with RVD($\mathcal{S}_S^r$) into RVD($\mathcal{S}_W^r \cup \mathcal{S}_S^r$).
Respectively RVD($\mathcal{S}_E^r \cup \mathcal{S}_N^r$).



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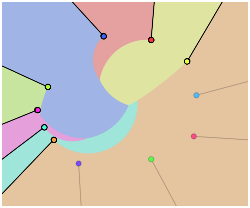
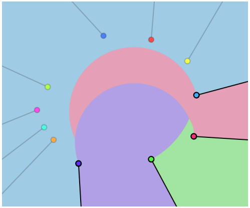
Respectively RVD($\mathcal{S}_E^r \cup \mathcal{S}_N^r$).

Step 3.b.

Merge RVD($\mathcal{S}_W^r \cup \mathcal{S}_S^r$) with RVD($\mathcal{S}_E^r \cup \mathcal{S}_N^r$) confined into P.
Obtain PRVD($\mathcal{S}_P$).





Step 3. Merging the 4 diagrams

Step 3.a.

Merge RVD($\mathcal{S}^r_W$) with RVD($\mathcal{S}^r_S$) into RVD($\mathcal{S}^r_W \cup \mathcal{S}^r_S$).
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Merge RVD($\mathcal{S}^r_W \cup \mathcal{S}^r_S$) with RVD($\mathcal{S}^r_E \cup \mathcal{S}^r_N$) confined into P.
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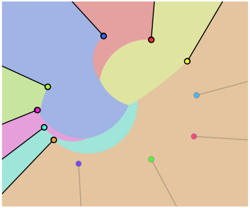
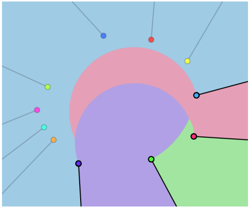
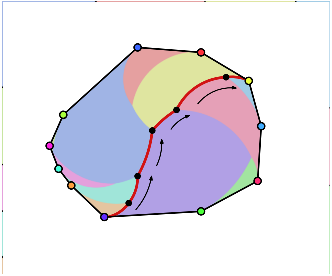

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Step 3.a.

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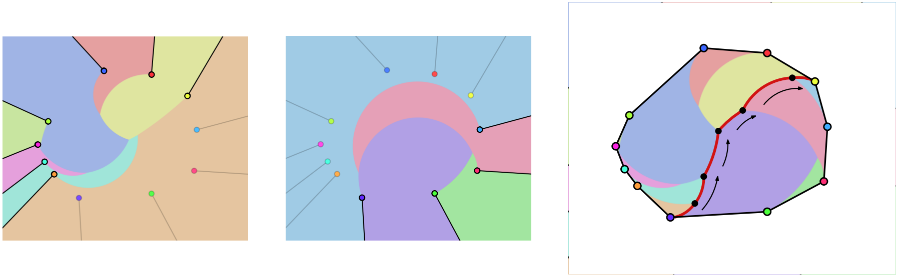
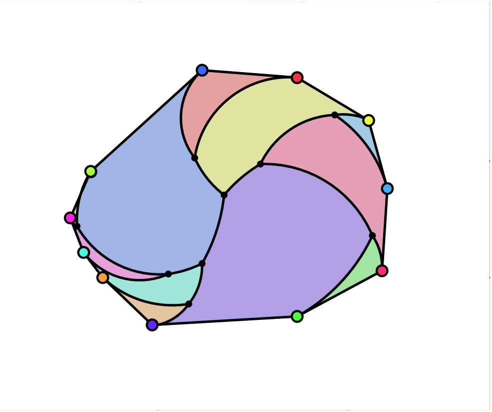


Finding the Brocard angle

Theorem
PRVD($\mathcal{S}_P$) *can be constructed* $\Theta(n)$ *time.*



Finding the Brocard angle

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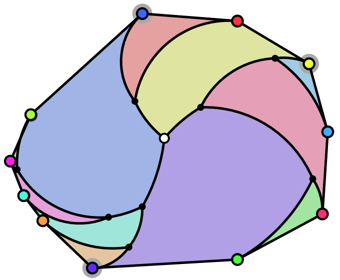

Brocard Illumination of P

1. Construct PRVD($\mathcal{S}_P$). $\Theta(n)$ time.
2. Traverse PRVD($\mathcal{S}_P$). $\Theta(n)$ time.

Finding the Brocard angle

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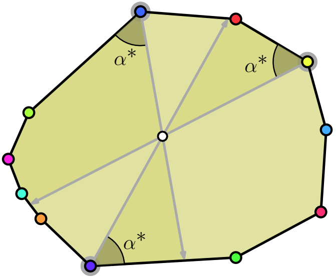

Brocard Illumination of P

1. Construct PRVD($\mathcal{S}_P$). $\Theta(n)$ time.
2. Traverse PRVD($\mathcal{S}_P$). $\Theta(n)$ time.

- Three α^*-floodlights suffice.

Finding the Brocard angle

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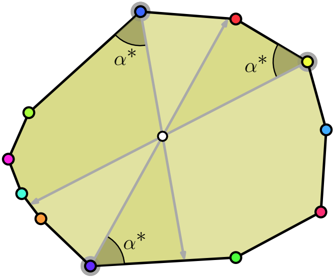

Brocard Illumination of P

1. Construct PRVD($\mathcal{S}_P$). $\Theta(n)$ time.
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- Three α^*-floodlights suffice.
- $\alpha^* \in (0, \pi/2 - \pi/n]$.

Summary and open questions

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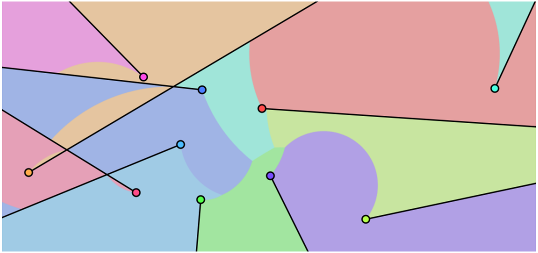
- RVD: definition, useful in Floodlight Illumination.
- RVD in $\mathbb{R}^2$: properties, complexity & algorithm.
- Brocard Illumination of convex polygons: optimal $\Theta(n)$ time.



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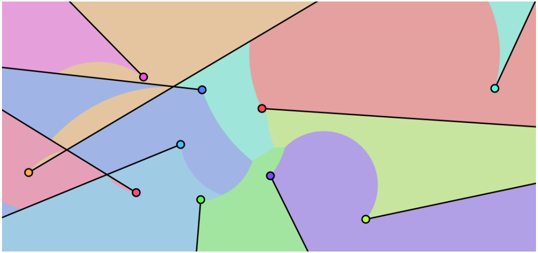

Open Questions

- Gap in the complexity of RVD($\mathcal{S}$) in $\mathbb{R}^2$: $\Omega(n^2) - O(n^{2+\epsilon})$
- Extend our approach to other classes of polygons.